A HEIGHT GAP THEOREM FOR COEFFICIENTS OF MAHLER FUNCTIONS

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Abstract. We study the asymptotic growth of coefficients of Mahler power series with algebraic coefficients, as measured by their logarithmic Weil height. We show that there are five different growth behaviors, all of which being reached. Thus, there are *gaps* in the possible growths. In proving this height gap theorem, we obtain that a k-Mahler function is k-regular if and only if its coefficients have height in $O(\log n)$. Moreover, we deduce that, over an arbitrary ground field of characteristic zero, a k-Mahler function is k-automatic if and only if its coefficients belong to a finite set. As a by-product of our results, we also recover a conjecture of Becker which was recently settled by Bell, Chyzak, Coons, and Dumas.

1. Introduction

The study of power series solutions to linear differential equations with coefficients in $\overline{\mathbb{Q}}[z]$ provides a deep interplay between various fields of mathematics and physics, including combinatorics and number theory. For instance, the study of generating series in enumerative combinatorics benefits from the useful dictionary between asymptotics of coefficients of D-finite power series and the type of singularities of the corresponding differential equation (see [FS09]). More surprisingly, prescribing some kind of arithmetic behavior for coefficients gives rise to powerful number theoretical consequences, as first perceived by Siegel [Sie29] when introducing E- and G-functions, and pursued more recently by André [And00a, And00b] in his study of arithmetic Gevrey series.

This paper deals with the arithmetic behavior of coefficients of *Mahler functions*, or *M-functions*, which are power series of a very different kind. Unless it is rational, an M-function never satisfies a linear or even an algebraic differential equation [ADH21]. Instead, M-functions are solutions to linear difference equations with coefficients in $\overline{\mathbb{Q}}[z]$ associated with the Mahler operator $z \mapsto z^k$, where $k \geq 2$ is a natural number. Precisely, a power series $f(z) \in \overline{\mathbb{Q}}[\![z]\!]$ is a *k-Mahler function*, or for short *k-Mahler*, if it satisfies an equation of the form

$$p_0(z)f(z) + \cdots + p_d(z)f(z^{k^d}) = 0 \tag{1}$$

with $p_0(z), \ldots, p_d(z) \in \overline{\mathbb{Q}}[z]$ and $p_0(z)p_d(z) \neq 0$. A power series is an M-function if it is a k-Mahler function for some k. The study of M-functions and their values was initiated at the end of the 1920's by Mahler [Mah29, Mah30a, Mah30b], who developed a new

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direction in transcendence theory, nowadays known as Mahler's method. In fact, Mahler only considered order one equations, but possibly inhomogeneous and also non-linear ones. The interest for M-functions of arbitrary order really took on a new significance at the beginning of the 1980s after Mendès France popularized among number theorists a result of Cobham [Cob68] stating that *automatic power series* are M-functions. After recent results [Phi15, AF17], the transcendence theory of M-functions mirrors exactly the one of E-functions. Beyond Mahler's method and automata theory, it is worth mentioning that M-functions naturally occur as generating functions in various other topics such as combinatorics of partitions, numeration, and analysis of algorithms. In particular, the *regular power series* introduced by Allouche and Shallit [AS92] form a distinguished class of M-functions. There is also a mysterious interplay between G-functions and M-functions that deserves more attention. Indeed, for some G-functions $\sum_{n=0}^{\infty} a_n z^n \in \mathbb{Q}[\![z]\!]$, the power series $\sum_{n=0}^{\infty} \mathsf{v}_p(a_n) z^n$, where $\mathsf{v}_p(a_n)$ is the p-adic valuation of a_n, turns out to be p-Mahler. This is likely related to the fact that Picard-Fuchs differential equations have a strong Frobenius structure for almost all primes. In recent years, there is renewed interest in M-functions, as evidenced by the flourishing literature on this topic. The latter includes discussions on various perspectives such as transcendence and algebraic independence, combinatorics and theoretical computer science, the study of Mahler's equations and associated Galois theories, and computational aspects. A number of references can be found in the survey [Ada19].

1.1. **The Height Gap Theorem.** Let us first recall that the coefficients of a k-Mahler function $\sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ satisfy some recurrence relation of the form

$$a_n = \sum_{j=1}^{s} -\alpha_j a_{n-j} + \sum_{i=1}^{d} \sum_{j=0}^{s} \beta_{i,j} a_{\frac{n-j}{k^i}} \, ,$$

where α_j and $\beta_{i,j}$ are algebraic numbers, and n is large enough (see Equation (5), in which one can achieve $m = 0$ using Lemma 3.1). It follows that the field extension of $\mathbb{Q}$ generated by all coefficients a_n is a number field. In the sequel, we will measure the coefficients of an M-function by their logarithmic Weil height.

1.1.1. *The logarithmic Weil height.* For a number field, we normalize the non-trivial absolute values as in [BG06]. Thus, for $\mathbb{Q}$ and p a prime we let $|p|_p = 1/p$; for the archimedean place of $\mathbb{Q}$ we use the usual absolute value. For K a number field and a place w of K extending a place v of $\mathbb{Q}$, let

$$|\alpha|_{\mathsf{w}} \coloneqq |\mathrm{N}_{K_{\mathsf{w}}/\mathbb{Q}_{\mathsf{v}}}(\alpha)|_{\mathsf{v}}^{1/[K:\mathbb{Q}]}.$$

Then the set of places M_K on K satisfies the product formula. For $\alpha \in \overline{\mathbb{Q}}$ the *logarithmic absolute Weil height* is defined by

$$h(\alpha) = \log \prod_{\mathsf{v} \in M_K} \max\{1, |\alpha|_{\mathsf{v}}\},$$

where K is any number field containing α. The value $h(\alpha)$ in this definition does not depend on the choice of such a number field K. For $a/b \in \mathbb{Q} \setminus \{0\}$ with $a \in \mathbb{Z}$, $0 \neq b \in \mathbb{Z}$, and $\gcd(a, b) = 1$,

$$h(a/b) = \log \max\{|a|, |b|\}.$$

For more properties about the logarithmic Weil height, as well as for comparison with other notions of height, we refer the reader to [Wal00, Chapter 3].

1.1.2. *Landau notation.* Let $(a_n)_{n\geq 0}$ be a sequence of nonnegative real numbers and $(b_n)_{n\geq 0}$ be a sequence of, eventually positive, real numbers. As usual, the notation $a_n \in O(b_n)$ means that there exists a positive number c such that $a_n < cb_n$ for every sufficiently large positive integer n, while the notation $a_n = o(b_n)$ means that a_n/b_n tends to zero as n tends to infinity. Moreover, sticking to the usual practice in number theory, we write $a_n \in \Omega(b_n)$ when $a_n \notin o(b_n)$, that is, when there exists a positive number c such that $a_n > cb_n$ for infinitely many positive integers n. We also write $a_n = O \cap \Omega(b_n)$ when both $a_n \in O(b_n)$ and $a_n \in \Omega(b_n)$.

We are now ready to state our first main result.

Theorem 1.1 (Height Gap Theorem)**.** *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be an M-function. Then one of the following properties holds.*

(1) $h(a_n) \in O \cap \Omega(n)$.
(2) $h(a_n) \in O \cap \Omega(\log^2 n)$.
(3) $h(a_n) \in O \cap \Omega(\log n)$.
(4) $h(a_n) \in O \cap \Omega(\log\log n)$.
(5) $h(a_n) \in O(1)$.

Theorem 1.1 implies that the coefficients of an M-function can only exhibit certain specific growth behaviors. For instance, as $h(a_n) \in o(n)$ forces $h(a_n) \in O(\log^2 n)$, there cannot be such a power series with $h(a_n) \sim \log^3 n$. Thus, there are *gaps* in the possible growths. Let us make few comments on Theorem 1.1.

- In Section 2, we provide the reader with examples for each of the five growth classes, thereby showing that all of them occur.
- There is no chance in general of replacing lower bounds of the type Ω by stronger ones. For instance, the 2-Mahler function $\sum_{n=0}^{\infty} 2^n z^{2^n}$ belongs to class (3), but most of its coefficients vanish.
- An M-function $f(z)$ can be uniquely specified by the finite data consisting of a k-Mahler equation it satisfies and sufficiently many initial coefficients of the power series. Assuming the knowledge of such data, we will show that it is decidable which of the five growth classes in Theorem 1.1 the function $f(z)$ falls into. This is Theorem 12.1.

1.2. **Height and structural properties of M-functions.** We already alluded to the fact that inside the ring of k-Mahler functions two subsets are usually distinguished, leading to the following hierarchy:

$$\{k\text{-automatic functions}\} \subsetneq \{k\text{-regular functions}\} \subsetneq \{k\text{-Mahler functions}\}.$$

We refer the reader to [AS03a] and Section 3 for precise definitions and more details about automatic and regular power series. The following result shows that each of these two subsets turns out to be equal to a special class in the refined hierarchy provided by Theorem 1.1.

Theorem 1.2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be a k-Mahler function. Then the two following properties hold.*

(a) *$f(z)$ is k-automatic if and only if $h(a_n) \in O(1)$, that is, if and only if the sequence a_n takes values in a finite set.*
(b) *$f(z)$ is k-regular if and only if $h(a_n) \in O(\log n)$.*

Case (a) of Theorem 1.2 extends to arbitrary ground fields of characteristic zero (see Theorem 11.1). This generalizes the well-known fact that k-regular sequences taking only finitely many values are k-automatic [AS03a, Theorem 16.1.5].

In fact, in proving Theorem 1.1, we will show that each of the five growth classes corresponds to natural structural properties of the k-Mahler equation, respectively, the coefficient series. The corresponding results are stated in Theorems 6.1, 7.1, 8.3 and 9.1. Theorem 1.2 above provides only a sample. In order to get such structural results, we reinforce the importance of measuring the size of coefficients by their height and not only by their modulus. For instance, all the following three Mahler functions

$$\prod_{n=0}^{\infty}(1 - z^{2^n}) \ , \ \sum_{n=0}^{\infty} 2^{-n} z^{2^n} \ , \ \frac{1}{1 - z/2} \cdot \prod_{n=0}^{\infty}(1 - z^{2^n})$$

have bounded rational coefficients, so we cannot distinguish them through the growth of their coefficients. This is a deficiency, for the first one is automatic, the second one is regular but not automatic, and the third one is not regular. However, their coefficients have different height growth behaviors and they can be distinguished by Theorem 1.1. They belong respectively to classes (1), (3), and (5).

The decidability of growth properties of k-regular sequences with respect to the usual (archimedean) absolute value has recently been considered by Krenn and Shallit [KS20]. In contrast to our results, many of these properties become undecidable. For instance, it is undecidable whether the sequence of coefficients is bounded [KS20, Theorem D].

Outline. This article is organized as follows. Section 2 provides the reader with a collection of examples, showing that each of the five growth classes in the height gap theorem actually occurs. In Sections 3 and 4, we give some background about Mahler equations, automatic and regular power series, and Mahler's method. Sections 5 to 9 are then devoted to the proof of Theorem 1.1. In fact, we prove the much more precise Theorems 6.1, 7.1, 8.3 and 9.1.

In Section 10, we discuss how our main results imply Becker's conjecture. In Section 11, we characterize those k-Mahler functions which are automatic over an arbitrary ground field of characteristic zero. In the final Section 12, we deal with the question of decidability in Theorem 1.1.

Notation. Throughout the paper, we use the following notation. We let $k \geq 2$ be a natural number. We let Σ_k denote the alphabet $\{0, 1, \dots, k-1\}$ and Σ_k^* denote the free monoid generated by Σ_k, with neutral element ε. Given a positive integer n, we set $\langle n \rangle_k := w_r w_{r-1} \cdots w_0$ for the canonical base-k expansion of n (written from most to least significant digit), which means that $n = \sum_{i=0}^{r} w_i k^i$ with $w_i \in \Sigma_k$ and $w_r \neq 0$. Note that by convention $\langle 0 \rangle_k := \varepsilon$. Conversely, if $w := w_0 \cdots w_r$ is a finite word over the alphabet Σ_k, we set $[w]_k := \sum_{i=0}^{r} w_{r-i} k^i$. We let $\mathcal{U} \subseteq \overline{\mathbb{Q}}$ denote the set of all roots of unity. For $0 \neq \zeta \in \overline{\mathbb{Q}}$, observe that there exists $j > 0$ with $\zeta^{k^j} = \zeta$ if and only if $\zeta \in \mathcal{U}$

and ζ has order coprime to k. We let $\mathcal{U}_k \subseteq \mathcal{U}$ denote the set of roots of unity whose order is not coprime with k.

2. Witnessing Examples

In this section, we provide examples of Mahler functions $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ for each of the five growth classes occurring in the height gap theorem. We recall that a rational power series is k-Mahler for all $k \geq 2$.

Examples in $(O \cap \Omega)(n)$. The upper bound $h(a_n) \in O(n)$ holds for every Mahler function, by Theorem 1.1. To give examples of Mahler functions whose coefficient sequence has growth in $(O \cap \Omega)(n)$, it therefore suffices to find examples whose coefficient series has growth in $\Omega(n)$.

(a) The coefficients of the rational function

$$\frac{1}{1-2z} = \sum_{n=0}^{\infty} 2^n z^n$$

have linear growth since $h(2^n) = n \log(2)$.

(b) Let a, $k \geq 2$ be integers and consider the transcendental infinite product

$$\prod_{n=0}^{\infty} \frac{1}{1 - az^{k^n}} = \sum_{n=0}^{\infty} a_n z^n.$$

This power series is k-Mahler and a_n is at least as large as the coefficient of z^n in $1/(1-az)$, that is $a_n \geq a^n$. Hence $h(a_n) \geq n \log(a)$.

(c) The previous example has poles at a^{-1/k^n} for all $n \geq 1$, but it can be refined to one that is analytic in the open unit disk of $\mathbb{C}$. Let a, $k \geq 2$ be integers and let us consider the infinite product

$$\prod_{n=0}^{\infty} \frac{1}{1 - a^{-1} z^{k^n}} = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{Q}[\![z]\!].$$

A partition of n into k-powers is an expression $n = j_1 k^{n_1} + \cdots + j_r k^{n_r}$ with $r \in \mathbb{Z}_{\geq 0}$, $0 \leq n_1 < \cdots < n_r$, and $j_1, \ldots, j_r \in \mathbb{Z}_{\geq 0}$. Expanding the factors in the definition of the infinite product as geometric series, we see that

$$a_n = \sum_{n = j_1 k^{n_1} + \cdots + j_r k^{n_r}} a^{-(j_1 + \cdots + j_r)},$$

where the sum is over partitions of n into k-powers. The partition $n = 1 + \cdots + 1 = n \cdot k^0$ gives a summand a^{-n}, and for all other summands $j_1 + \cdots + j_r < n$. Let p be a prime divisor of a. Then $|a_n|_p \geq p^n$, and therefore $h(a_n) \geq n \log p$.

Examples in $(O \cap \Omega)(\log^2 n)$. The following example is typical of the Mahler functions in the class $(O \cap \Omega)(\log^2 n)$. It will play a prominent role in Section 7.

(d) Let $k \geq 2$ be an integer and consider the infinite product of cyclotomic polynomials
$$\prod_{n=0}^{\infty} \frac{1}{1 - z^{k^n}} = \sum_{n=0}^{\infty} a_n z^n \,.$$
As in (c) above, we see that the integer a_n is equal to the number of partitions of n into k-powers. The asymptotics of the coefficient sequence a_n were first studied by Mahler [Mah40], who proved that
$$\log a_n \sim \frac{\log^2 n}{2 \log k} \,.$$
These results of Mahler have been refined and generalized by de Bruijn [dB48] and most recently by Dumas and Flajolet [DF96].

(e) Multiplying the infinite product in (d) by any nonzero k-regular power series with positive coefficients provides a transcendental k-Mahler function with the required growth behavior. In fact, Theorem 3.8 shows that examples in the class $(O \cap \Omega)(\log^2 n)$ are essentially all of that type.

Examples in $(O \cap \Omega)(\log n)$. For every regular sequence $(a_n)_{n \geq 0}$, the generating series $\sum_{n=0}^{\infty} a_n z^n$ is an M-function, and examples for which $h(a_n) \in (O \cap \Omega)(\log n)$ abound. We give some examples and refer the reader to [AS03a, Chapter 16.5] and [AS92, AS03b] for more.

(f) The rational power series
$$\frac{z}{(1-z)^2} = \sum_{n=0}^{\infty} n z^n \,,$$
falls into this class, since $h(n) = \log(n)$. More generally, if $p(z)$ is a non-constant polynomial with integer coefficients, then $\sum_{n=0}^{\infty} p(n) z^n$ is a rational function with the required growth behavior.

(g) Let $\mathsf{v}_p(n)$ denote the p-adic valuation of the natural number n, where p is a prime number. The power series
$$\sum_{n=0}^{\infty} \mathsf{v}_p(n!) z^n,$$
is p-regular [AS92, Example 8]. Moreover, by Legendre's formula $\mathsf{v}_p(n!) \sim n/(p-1)$.

(h) Let ℓ_n denote the number of positive integers at most equal to n that can be written as sum of three squares. The power series $\sum_{n=0}^{\infty} \ell_n z^n$ is 2-regular [AS03a, Example 16.5.2]. Since every integer not of the form $4^a(8b+7)$ can be written as a sum of three squares, the sequence has the required growth behavior.

(i) Any linear representation (u, μ, v) on the alphabet Σ_k gives rise to a k-regular sequence (see Definition 3.4). From our results, we will see that whenever there exists a word $w \in \Sigma_k^*$ such that the matrix $\mu(w)$ has an eigenvalue that is neither 0 nor a root of unity, then the sequence associated with this linear representation has the required growth behavior.

Examples in $(O\cap\Omega)(\log\log n)$. M-functions whose coefficients have growth in $(O\cap\Omega)(\log\log n)$ can again be found by looking at generating functions of suitable k-regular sequences.

(j) The power series
$$\sum_{n=1}^{\infty}(1+\lfloor\log_2 n\rfloor)z^n$$
is 2-regular [AS92, Example 11], and clearly has the required growth behavior.

(k) Let s_n denote the sum of digits in the base-k expansion of n. Then clearly $(s_n)_{n\geq 0}$ is k-regular, and therefore the power series $\sum_{n=0}^{\infty}s_n z^n$ is k-Mahler. Since $s_n\in O(\log n)$, it holds that $h(s_n)\in O(\log\log n)$. Moreover, for $e\geq 0$ and $n=k^e-1$ we have $s_n=(k-1)e\sim(k-1)\log_k n$. Hence s_n has he required growth behavior.

Examples in $O(1)$. By Theorem 1.2, this class of M-functions corresponds exactly to generating series of automatic sequences. We refer the reader to the monograph [AS03a] for numerous examples, including the generating series of the Thue-Morse sequence, the Rudin-Shapiro sequence, the Baum-Sweet sequence, and the paperfolding sequence, to name a few.

3. Preliminaries

Throughout this section, we let K be a field. We will later restrict ourselves to $K=\overline{\mathbb{Q}}$. We recall k-Mahler, k-automatic, k-regular, and k-Becker power series and their relation to each other.

3.1. **Mahler functions, equations, and systems.** Let us recall that a power series $f(z)\in K[\![z]\!]$ is a k-Mahler function if it satisfies an equation of the form (1), that is if there exist a nonnegative integer d and polynomials $p_0(z),\dots,p_d(z)\in K[z]$, not all zero, such that
$$p_0(z)f(z)+p_1(z)f(z^k)+\cdots+p_d(z)f(z^{k^d})=0\,.$$
It can be shown that every Mahler function satisfies such a functional equation with $p_0(z)p_d(z)\neq 0$ and $p_0(z)$, …, $p_d(z)$ coprime [AB17, Lemma 4.1]. As we will only be interested in the asymptotic behavior of the coefficients, the following lemma allows a further simplification of the Mahler equation.

Lemma 3.1. *Suppose $f(z)=\sum_{n=0}^{\infty}a_n z^n\in K[\![z]\!]$ satisfies a Mahler equation*
$$p_0(z)f(z)=p_1(z)f(z^k)+\cdots+p_d(z)f(z^{k^d}) \tag{2}$$
with $p_0(z)p_d(z)\neq 0$ and $p_0(z),\dots,p_d(z)$ coprime. Then there exists $n_0\geq 0$ such that $a_{n_0}\neq 0$ and $f_0(z):=\sum_{n=0}^{\infty}a_{n+n_0}z^n$ satisfies a k-Mahler equation
$$q_0(z)f_0(z)=q_1(z)f_0(z^k)+\cdots+q_{d+1}(z)f_0(z^{k^{d+1}})$$
with polynomials $q_0(z),\dots,q_{d+1}(z)$ satisfying the following conditions.

(i) *One has $q_0(0)=1$.*
(ii) *If $0\neq\lambda\in K$, then $p_0(\lambda)=0$ implies $q_0(\lambda)=0$.*
(iii) *If $0\neq\zeta\in K$ with $p_0(\zeta)=0$ and $\zeta^k=\zeta$, then $q_i(\zeta)\neq 0$ for some $i\in\{1,\dots,d+1\}$.*

Moreover, if $f(z)$ has at least two nonzero coefficients, then $f_0(z)$ is non-constant.

Proof. By [AB17, Lemma 6.1]; the final statement requires an inspection of the proof. □

We also need the following fact, a more general version of which is, for instance, proved in [AB17, Proposition 8.1].

Lemma 3.2. *If $f(z) \in K[\![z]\!]$ is k-Mahler and e is a positive integer, then $f(z)$ is also k^e-Mahler.*

3.1.1. *Linear Mahler systems.* A power series $f(z) \in K[\![z]\!]$ is k-Mahler if and only if it satisfies a *linear k-Mahler system*. That is, there exist $f_1(z) := f(z), \dots, f_d(z) \in K[\![z]\!]$ and $A(z) \in \mathrm{GL}_d(K(z))$ such that

$$\begin{pmatrix} f_1(z) \\ \vdots \\ f_d(z) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z^k) \\ \vdots \\ f_d(z^k) \end{pmatrix}. \tag{3}$$

Indeed, given $f(z)$ satisfying a k-Mahler equation $f(z) = r_1(z)f(z^k) + \dots + r_d(z)f(z^{k^d})$ with $r_1(z), \dots r_d(z) \in K(z)$ and $r_d(z) \neq 0$, the vector

$$\left(f(z), \dots, f(z^{k^{d-1}})\right)^T$$

satisfies an equation of the form (3) with $A(z)$ a companion matrix. Conversely, iterating an equation of the form (3), and using the invertibility of $A(z)$, it follows that each $f_i(z^{k^j})$ is contained in the finite-dimensional $K(z)$-vector space spanned by $f_1(z), \dots, f_d(z)$. Hence the power series $f_1(z^{k^j})$, $j \geq 0$, are linearly dependent over $K[z]$.

3.1.2. *Analytic properties.* Let us assume that $K = \overline{\mathbb{Q}}$. If $f(z) \in \overline{\mathbb{Q}}[\![z]\!]$ is a k-Mahler function, then there exists a number field K with $f(z) \in K[\![z]\!]$. This is so because all sufficiently high coefficients of $f(z)$ are determined recursively by lower ones (see [Dum93, Chapitre 3.2.2] or [AF18]). Let v be a place of K and $|\cdot|_\mathsf{v}$ be an absolute value associated with v. We let K_v denote the completion of K with respect to the absolute value $|\cdot|_\mathsf{v}$. We also let C_v denote the completion of the algebraic closure of K_v and $\overline{K}$ the algebraic closure of K in C_v. Recall that C_v is both algebraically closed and complete. The power series $f(z)$ is analytic in a neighborhood of 0 in C_v (see, for instance, [Dum93, Chapitre 3.3]). The Mahler equation then implies that $f(z)$ is meromorphic in the open unit disk $B_{|\cdot|_\mathsf{v}}(0,1)$ in C_v.

3.2. **Automatic and regular power series.** We recall the notion of k-automatic and k-regular sequences. For more background see Allouche and Shallit [AS03a] or Berstel and Reutenauer [BR11, Chapter 5].

A sequence $(a_n)_{n\geq 0}$ is *k-automatic* if there exists a finite automaton that, given as input the base k representation of n, reaches an output state labeled by a_n. Equivalently, the sequence $(a_n)_{n\geq 0}$ is k-automatic if and only if its k-kernel is a finite set.

Definition 3.3. Let $\mathbf{a} := (a_n)_{n\geq 0}$ be a sequence with values in a set S. The *k-kernel* of $\mathbf{a}$ is

$$\{\, (a_{k^e n + r})_{n\geq 0} : e \in \mathbb{Z}_{\geq 0},\ 0 \leq r \leq k^e - 1 \,\}.$$

Let us now restrict to sequences taking values in the field K. Then a sequence $(a_n)_{n\geq 0}$ is said to be *k-regular* if its k-kernel is contained in a finite-dimensional vector subspace of the K-vector space of all K-valued sequences. Obviously k-automatic sequences are k-regular. A k-regular sequence is k-automatic if and only if it takes only finitely many values [AS03a, Theorem 16.1.5]. There are several other characterizations of k-regular sequences [AS03a, Theorems 16.1.3 and 16.2.3]. We recall one characterization that will be essential.

Definition 3.4. A *linear representation* on the alphabet Σ_k is a triple (u, μ, v) where $u \in K^{1\times d}$, $v \in K^{d\times 1}$, and $\mu\colon \Sigma_k^* \to K^{d\times d}$ is a monoid homomorphism ($d \in \mathbb{Z}_{\geq 0}$). The linear representation is *minimal* if the dimension d is minimal amongst all $d' \geq 0$ and d'-dimensional linear representations (u', μ', v') such that $u\mu(w)v = u'\mu'(w)v'$ for all $w \in \Sigma_k^*$. Equivalently, $u\mu(\Sigma_k^*)$ spans $K^{1\times d}$ and $\mu(\Sigma_k^*)v$ spans $K^{d\times 1}$.

Theorem 3.5. *Let $(a_n)_{n\geq 0}$ be a sequence taking values in K. The following statements are equivalent.*

(a) *The sequence $(a_n)_{n\geq 0}$ is k-regular.*
(b) *There exists a (minimal) linear representation (u, μ, v) on the alphabet Σ_k such that $a_{[w]_k} = u\mu(w)v$ for all words $w \in \Sigma_k^*$.*

Proof. The result is proved in [AS03a, Theorem 16.2.3]. □

A power series $f(z) = \sum_{n=0}^{\infty} a_n z^n \in K[\![z]\!]$ is said to be *k-automatic*, respectively *k-regular* if the sequence $(a_n)_{n\geq 0}$ is k-automatic, respectively k-regular.

3.3. **Becker power series.** The connection between k-regular sequences in the sense of Allouche and Shallit and coefficients of k-Mahler power series was studied by Becker, who proved that k-regular power series are k-Mahler [Bec94, Theorem 1]. He also showed that the converse is false in general: a k-Mahler power series need *not* be k-regular [Bec94, Proposition 1]. However, he did obtain a partial converse. This motivates the next definition.

Definition 3.6. A power series $f(z) \in K[\![z]\!]$ is a *k-Becker function* (or, in short, *k-Becker*) if there exist a positive integer d and polynomials $p_1(z), \dots, p_d(z) \in K[z]$, not all zero, such that

$$f(z) = p_1(z)f(z^k) + \cdots + p_d(z)f(z^{k^d}).$$

Theorem 3.7 ([Bec94, Theorem 2])**.** *If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ is a k-Becker power series, then it is k-regular.*

In view of these results, one may also ask for a precise characterization of k-regular power series in terms of k-Becker power series. This gives rise to a conjecture of Becker, recently settled in [BCCD19], and discussed in Section 10. For Mahler functions, there exists the following useful decomposition due to Dumas.

Theorem 3.8 ([Dum93, Théorème 31, p.153])**.** *Let $f(z) \in K[\![z]\!]$ be k-Mahler satisfying an equation*

$$p_0(z)f(z) + p_1(z)f(z^k) + \cdots + p_d(z)f(z^{k^d}) = 0$$

with $p_0(z), \ldots, p_d(z) \in K[z]$ *and* $p_0(0) = 1$. *Then*

$$f(z) = \frac{g(z)}{\prod_{i=0}^{\infty} p_0(z^{k^i})}\,,$$

where $g(z) \in K[\![z]\!]$ *is a* k*-Becker power series.*

3.4. **The Mahler denominator.** As is already hinted at by Becker's result, the polynomial $p_0(z)$ in a Mahler equation (1) will play a prominent role in our arguments. This prompts the following definition.

Definition 3.9. Let $f(z) \in K[\![z]\!]$ be a k-Mahler power series, and let

$$\mathfrak{I} = \left\{\, p(z) \in K[z] : p(z)f(z) \in \sum_{i=1}^{\infty} K[z]f(z^{k^i}) \,\right\}.$$

The k*-Mahler denominator* of $f(z)$ is the unique generator $\mathfrak{d}(z) \in K[z]$ of the ideal $\mathfrak{I}$, with the lowest nonzero coefficient of $\mathfrak{d}(z)$ being 1.

Since $K[z]$ is a principal ideal domain, there indeed exists such a generator. Observe that $f(z)$ is k-Becker if and only if $\mathfrak{d}(z) \equiv 1$. It is tempting to hope that the k-Mahler denominator is equal to the polynomial $p_0(z)$ in the minimal k-Mahler equation, that is the equation

$$p_0(z)f(z) + p_1(z)f(z^k) + \cdots + p_d(z)f(z^{k^d}) = 0$$

with $p_0(z)p_d(z) \neq 0$, minimal d, and coprime $p_0(z), \ldots, p_d(z)$. While this is often the case, in general this is *not* so. See Example 3.10 for a counterexample. By definition $\mathfrak{d}(z)$ divides $p_0(z)$. It is tempting to hope that, to determine the types of roots of $\mathfrak{d}(z)$, it suffices to consider those of $p_0(z)$. Unfortunately, this hope is also thwarted by the following example.

Example 3.10. The equation

$$(z - 1/2)f(z) - (z - 1/8)(z^3 - 1/2)f(z^3) = 0$$

has only one nonzero solution (up to a scalar) and is minimal with respect to this solution. However, this solution is k-regular because

$$f(z) = (z - 1/8)(z^2 + 1/2z + 1/4)(z^9 - 1/2)f(z^9)\,.$$

The expected pole at $1/2$ disappears after one iteration of the equation.

We will see with Theorems 6.1 and 7.1 that locating the roots of the k-Mahler denominator provides a characterization of those k-Mahler functions with $h(a_n) \in O(\log^2 n)$ and with $h(a_n) \in O(\log n)$. However, given a Mahler function with $h(a_n) \in O(\log n)$, its Mahler denominator is irrelevant in determining whether $h(a_n) \in O(\log\log n)$ or $h(a_n) \in O(1)$. For instance, all the three following 2-regular functions

$$\prod_{n=0}^{\infty}(1 - z^{2^n}) \;\;,\;\; \sum_{n=0}^{\infty} b_n z^n \;\;,\;\; \left(\frac{1}{1-z}\right)^2 \;\;,$$

where we let b_n denote the number of 0's in the binary expansion of n (with $b_0 = 1$), have a trivial Mahler denominator (*i.e.*, $\mathfrak{d}(z) = 1$). However, their coefficients have height in $O(1)$, $O \cap \Omega(\log\log n)$, and $O \cap \Omega(\log n)$, respectively.

4. Background about Mahler's method

Let us consider a linear k-Mahler system:

$$\begin{pmatrix} f_1(z) \\ \vdots \\ f_d(z) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z^k) \\ \vdots \\ f_d(z^k) \end{pmatrix} \tag{4}$$

where $A(z)$ is a matrix in $\mathrm{GL}_d(\overline{\mathbb{Q}}(z))$ and $f_1(z), \ldots, f_d(z) \in \overline{\mathbb{Q}}[\![z]\!]$. There exists a number field K such that the $f_i(z)$'s belong to $K[\![z]\!]$ and $A(z) \in \mathrm{GL}_d(K(z))$. Let v be a place of K and $|\cdot|_{\mathsf{v}}$ be an absolute value associated with v. As before, we let K_{v} denote the completion of K with respect to the absolute value $|\cdot|_{\mathsf{v}}$. We also let C_{v} denote the completion of the algebraic closure of K_{v} and $\overline{K}$ the algebraic closure of K in C_{v}.

Definition 4.1. A point $\alpha \in C_{\mathsf{v}}$ is called *singular* with respect to (4) if there exists a nonnegative integer n such that α^{k^n} is a pole of one of the coefficients of the matrix $A(z)$ or of the matrix $A^{-1}(z)$. We say that α is *regular* otherwise, that is, α is regular if both $A(\alpha^{k^n})$ and $A^{-1}(\alpha^{k^n})$ are well-defined for every nonnegative integer n.

We recall that the power series $f_1(z), \ldots, f_d(z)$ are meromorphic in the open unit disc of C_{v} and analytic in some neighborhood of the origin. Moreover, if α is a regular point such that $|\alpha|_{\mathsf{v}} < 1$, then the functions $f_1(z), \ldots, f_d(z)$ are well-defined at α. We also recall that given a field K, and elements $a_1, \ldots, a_m$ in some field extension of K, the notation $\mathrm{tr.deg}_K(a_1, \ldots, a_m)$ stands for the transcendence degree over K of the field extension $K(a_1, \ldots, a_m)$.

Theorem 4.2. *Let $f_1(z), \ldots, f_d(z) \in K[\![z]\!]$ be solutions to (4). Let $\alpha \in \overline{K}$, $0 < |\alpha|_{\mathsf{v}} < 1$ be a regular point with respect to this system. Then*

$$\mathrm{tr.deg}_{\overline{K}}(f_1(\alpha), \ldots, f_d(\alpha)) = \mathrm{tr.deg}_{\overline{K}(z)}(f_1(z), \ldots, f_d(z))\,.$$

In the case where $|\cdot|_{\mathsf{v}}$ is the usual absolute value on $\mathbb{C}$, this classical result is due to Nishioka [Nis90]. The proof of Nishioka is based on some techniques from commutative algebra introduced in the framework of algebraic independence by Nesterenko in the late Seventies. Recently, Fernandes [Fer18] observed that Theorem 4.2 can also be deduced from a general algebraic independence criterion due to Philippon [Phi86, Phi92]. This allows her to extend Nishioka's theorem in the framework of function fields of positive characteristic. Using the fact that the criteria obtained by Philippon also apply to any absolute value associated with a place of a number field (see for instance Theorem 2.11 in [Phi86]), we can argue exactly as in the proof of Theorem 1.3 of [Fer18] to prove Theorem 4.2.

Theorem 4.3. *Let $f_1(z), \ldots, f_d(z) \in K[\![z]\!]$ be solutions to (4). Let $\alpha \in \overline{K}$, $0 < |\alpha|_{\mathsf{v}} < 1$ be a regular point for this system. Then for all homogeneous polynomials $P(X_1, \ldots, X_d) \in \overline{K}[X_1, \ldots, X_d]$ such that*

$$P(f_1(\alpha), \ldots, f_d(\alpha)) = 0\,,$$

there exists $Q(z, X_1, \ldots, X_d) \in \overline{K}[z, X_1, \ldots, X_d]$, homogeneous in $X_1, \ldots, X_d$, such that

$$Q(z, f_1(z), \ldots, f_d(z)) = 0$$

and

$$Q(\alpha, X_1, \ldots, X_d) = P(X_1, \ldots, X_i).$$

Proof. In the case where $|\cdot|_{\mathsf{v}}$ is the usual absolute value on $\mathbb{C}$, this result is due to Adamczewski and Faverjon in [AF17, Theorem 1.4]. It is obtained as a consequence of the main result of Philippon in [Phi15], which itself is based on Nishioka's theorem. The strategy to deduce this result from Nishioka's theorem is detailed in [AF17], see Proposition 3.1. The arguments are based on basic facts from commutative algebra that also apply to our more general framework. The two main ingredients that we have to be careful about are the following ones.

(i) A result by Krull saying that if $\mathfrak{p}$ is a homogeneous ideal in $K[z, X_0, \ldots, X_d]$ that is absolutely prime, then for all but finitely many $\alpha \in K$, the ideal $\mathrm{ev}_\alpha(\mathfrak{p})$ is a prime ideal of $K[X_0, \ldots, X_d]$. Here, we let $\mathrm{ev}_\alpha \colon K[z] \mapsto K$ denote the evaluation map at $z = \alpha$. See [Kru48].
(ii) The fact that the field extension $L \coloneqq \overline{K}(z)(f_1(z), \ldots, f_d(z))$ is regular, which means that an element of L is algebraic over $\overline{K}(z)$ if and only if it belongs to $\overline{K}(z)$.

We can use (i) in our framework for Krull proved his result for any base field K. To prove that (ii) also holds true in our framework, we need to know that a k-Mahler function in $\overline{K}[\![z]\!]$ is either rational or transcendental over $\overline{K}(z)$. There are several proofs for this result. For instance, Theorem 5.1.7 in [Nis96] provides a proof in the case where K is any field of characteristic 0. Then we can argue exactly as in the proof of Lemma 3.2 in [AF17] to deduce that the field extension $\overline{K}(z)(f_1(z), \ldots, f_d(z))$ is regular. ☐

As a corollary of Theorem 4.3, we deduce the following result.

Corollary 4.4. *Let $f_1(z), \ldots, f_d(z) \in K[\![z]\!]$ be solutions to (4). Let us assume that $f_1(z), \ldots, f_d(z)$ are linearly independent over $\overline{K}(z)$. Then there exists $0 < r < 1$, such that for every $\alpha \in \overline{K}$ with $0 < |\alpha|_{\mathsf{v}} < r$, the numbers $f_1(\alpha), \ldots, f_d(\alpha)$ are well-defined and linearly independent over $\overline{K}$.*

Proof. We first observe that if r is small enough, then α is a regular point with respect to (4) and the numbers $f_1(\alpha), \ldots, f_d(\alpha)$ are thus well-defined. Then the result follows directly from Theorem 4.3. ☐

5. Generic upper bound

To prove (1) of Theorem 1.1, giving a general upper bound on $h(a_n)$ for a Mahler function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$, we use a classical recursion for the sequence $(a_n)_{n\geq 0}$ that is deduced from the Mahler equation. Since this is somewhat lengthy and the proof of the upper bound for $h(a_n)$ in case (2) of Theorem 1.1, where we assume $h(a_n) \in o(n)$, works similarly, we establish both these bounds at the same time.

We need the following lemma. For archimedean absolute values, a more general result for finitely generated semigroups of matrices can be found in [Bel05].

Lemma 5.1. *Let d be a positive integer. Let $|\cdot|$ be an absolute value on $\overline{\mathbb{Q}}$, and let $\|\cdot\|$ be an operator norm on $\overline{\mathbb{Q}}^{d\times d}$ with respect to $|\cdot|$. Let $A \in \overline{\mathbb{Q}}^{d\times d}$ be a matrix such that*

$|\lambda| \le 1$ *for every eigenvalue* λ *of* A. *Then*

$$\|A^n\| \in \begin{cases} O(n^{d-1}) & \text{if } |\cdot| \text{ is archimedean,} \\ O(1) & \text{if } |\cdot| \text{ is non-archimedean.} \end{cases}$$

Proof. It suffices to show the claim for a Jordan block $\lambda + N \in \overline{\mathbb{Q}}^{s\times s}$ where $s \le d$, where $|\lambda| \le 1$, and where N is the $s \times s$-matrix with ones on the superdiagonal and zeroes everywhere else. Then N^i is the matrix that has ones on the i-th superdiagonal and zeroes everywhere else, with $N^i = 0$ for $i \ge s$. Thus

$$(\lambda + N)^n = \sum_{i=0}^{s-1} \binom{n}{i} \lambda^{n-i} N^i \qquad \text{for } n \ge s.$$

Now $\|(\lambda + N)^n\| \le C\left|\binom{n}{s-1}\right|$ for some constant C, and the claim follows. ☐

Proposition 5.2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ *be a* k*-Mahler function. Then the following properties hold.*

1. *One has* $h(a_n) \in O(n)$.
2. *Suppose in addition that all roots of the* k*-Mahler denominator of* $f(z)$ *are contained in* $\{0\} \cup \mathcal{U}$. *Then* $h(a_n) \in O(\log^2 n)$.

Proof. We may assume $f(z) \ne 0$. The series $f(z)$ satisfies a k-Mahler equation

$$p_0(z)f(z) = p_1(z)f(z^k) + \cdots + p_d(z)f(z^{k^d})$$

with $p_0(z) = \mathfrak{d}(z)$ the k-Mahler denominator and $d \ge 1$. Extend the sequence a_n to rational indices by setting $a_r = 0$ for all $r \in \mathbb{Q} \setminus \mathbb{Z}_{\ge 0}$. Let $s = \max\{\deg p_i(z) : i \in 0, \dots, d\}$. Let $p_0(z) = z^m(1 + \alpha_1 z + \cdots + \alpha_{s-m}z^{s-m})$ with $\alpha_1, \dots, \alpha_{s-m} \in \overline{\mathbb{Q}}$, and, for $i \in \{1, \dots, d\}$, let $q_i(z) = \sum_{j=0}^{s} \beta_{i,j} z^j$ with $\beta_{i,j} \in \overline{\mathbb{Q}}$. Comparing coefficients in the Mahler equation, we have

$$a_{n-m} + \sum_{j=1}^{s-m} \alpha_j a_{n-m-j} = \sum_{i=1}^{d} \sum_{j=0}^{s} \beta_{i,j} a_{\frac{n-j}{k^i}} \qquad \text{for } n \in \mathbb{Z}.$$

Shifting the indices by m, we obtain

$$a_n = \sum_{j=1}^{s-m} -\alpha_j a_{n-j} + \sum_{i=1}^{d} \sum_{j=0}^{s} \beta_{i,j} a_{\frac{n+m-j}{k^i}} \qquad \text{for } n \in \mathbb{Z}. \tag{5}$$

If $n > m$, then $n > (n+m)/2 \ge (n+m-j)/k^i$ for $i \ge 1$ and $j \ge 0$. Thus, Eq. (5) allows the recursive computation of a_n for $n > m$ from $a_0, \dots, a_m$. We now write this as a matrix equation. For $i \in \{0, \dots, d\}$, let

$$\mathbf{a}_i(n) := \begin{pmatrix} a_{n/k^i} \\ a_{(n-1)/k^i} \\ \vdots \\ a_{(n-s)/k^i} \end{pmatrix}.$$

Let A, B_1, …, $B_d \in \overline{\mathbb{Q}}^{(s+1)\times(s+1)}$ be given by

$$B_i = \begin{pmatrix} \beta_{i,0} & \beta_{i,1} & \dots & \beta_{i,s} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -\boldsymbol{\alpha} & 0 \\ I_{s\times s} & 0_{s\times 1} \end{pmatrix},$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{s-m}, 0, \dots, 0) \in \overline{\mathbb{Q}}^{1\times s}$, where $I_{s\times s}$ is the $s \times s$ identity matrix, and $0_{s\times 1}$ is the $s \times 1$ matrix containing zeroes. The characteristic polynomial of A is $z^{s+1} + \alpha_1 z^s + \dots + \alpha_{s-m} z^{m+1} = z^{m+s+1} p_0(1/z) \in \overline{\mathbb{Q}}[z]$.

Now

$$\mathbf{a}_0(n) = A\mathbf{a}_0(n-1) + \sum_{i=1}^{d} B_i \mathbf{a}_i(n+m) \qquad \text{for } n > m.$$

Fix $n_0 > m$. Recursively substituting for $\mathbf{a}_0(n-j)$, for $n \geq n_0$, we get

$$\mathbf{a}_0(n) = A^{n-n_0}\mathbf{a}_0(n_0) + \sum_{j=0}^{n-n_0-1} \sum_{i=1}^{d} A^j B_i \mathbf{a}_i(n+m-j).$$

The recursion formula for $(a_n)_{n\geq 0}$ implies that there is a number field K containing all α_i, $\beta_{i,j}$ and a_n for $n \geq 0$. For each place v of K, let $|\cdot|_{\mathsf{v}}$ be the corresponding absolute value and let $\|\cdot\|_{\mathsf{v}}$ be the induced maximum norm. We also write $\|\cdot\|_{\mathsf{v}}$ for the operator norm on $K^{(s+1)\times(s+1)}$. Let $\varepsilon_{\mathsf{v}}(n) = n$ if v is archimedean, and $\varepsilon_{\mathsf{v}}(n) = 1$ if v is non-archimedean. Then

$$\|\mathbf{a}_0(n)\|_{\mathsf{v}} \leq \varepsilon_{\mathsf{v}}(dn) \cdot \max\Big\{ \|A^{n-n_0}\|_{\mathsf{v}} \cdot \|\mathbf{a}_0(n_0)\|_{\mathsf{v}},\ \|A^j\|_{\mathsf{v}} \cdot \|B_i\|_{\mathsf{v}} \cdot \|\mathbf{a}_i(n+m-j)\|_{\mathsf{v}} \Big\} \tag{6}$$

where $i \in \{1, \dots, d\}$ and $j \in \{0, \dots, n-n_0-1\}$. Let S be the *finite* set consisting of all places v that are archimedean or for which $\|A\|_{\mathsf{v}} > 1$, or $|a_n|_{\mathsf{v}} > 1$ for some $n \in \{0, \dots, n_0\}$, or $\|B_i\|_{\mathsf{v}} > 1$ for some $i \in \{1, \dots, d\}$. Note that, for $\mathsf{v} \notin S$, also $\|A^n\|_{\mathsf{v}} \leq \|A\|_{\mathsf{v}}^n \leq 1$ for all $n \geq 1$. If $\mathsf{v} \notin S$, then, by induction, Eq. (6) implies $|a_n|_{\mathsf{v}} \leq \|\mathbf{a}_0(n)\|_{\mathsf{v}} \leq 1$ for all $n \geq n_0$. Therefore

$$h(a_n) = \log \prod_{\mathsf{v}\in S} \max\{1, |a_n|_{\mathsf{v}}\}.$$

To show the claims, it suffices to obtain suitable bounds on $|a_n|_{\mathsf{v}}$ for $\mathsf{v} \in S$. We first prove the bound in (1).

Let $\mathsf{v} \in S$. We show $|a_n|_{\mathsf{v}} \leq c^n$ for some $c \in \mathbb{R}_{\geq 1}$ and all $n \geq 1$. First enlarge $n_0 > m$ and pick $c_0 \in \mathbb{R}_{\geq 1}$ so that $dnc^{1+m/2} \leq c^{n/6}$ for all $c \in \mathbb{R}_{\geq c_0}$ and $n \geq n_0$. Let $c \in \mathbb{R}_{\geq c_0}$ be sufficiently large so that $|a_n|_{\mathsf{v}} \leq c$ for all $n \in \{1, \dots, n_0\}$ and so that $\|B_i\|_{\mathsf{v}} \leq c$ for all $i \in \{1, \dots, d\}$. Enlarging c further, also suppose $\|A\|_{\mathsf{v}} \leq c^{1/3}$, so that $\|A^n\|_{\mathsf{v}} \leq c^{n/3}$.

We proceed by induction on n. For $1 \leq n \leq n_0$ the claim is true by choice of c. For $n > n_0$, Eq. (6) gives

$$\|\mathbf{a}_0(n)\|_{\mathsf{v}} \leq \varepsilon_{\mathsf{v}}(dn) \cdot c^{(n-n_0)/3} \cdot \max\Big\{ \|\mathbf{a}_0(n_0)\|_{\mathsf{v}},\ \|B_i\|_{\mathsf{v}} \cdot \|\mathbf{a}_i(n+m-j)\|_{\mathsf{v}} \Big\},$$

where $i \in \{1, \dots, d\}$ and $j \in \{0, \dots, n-n_0-1\}$. By induction hypothesis we can estimate $\|\mathbf{a}_i(n+m-j)\|_{\mathsf{v}} \le c^{(n+m)/k} \le c^{(n+m)/2}$, and therefore

$$\|\mathbf{a}_0(n)\|_{\mathsf{v}} \le dn \cdot c^{(n-n_0)/3} \cdot c \cdot c^{(n+m)/2} = dnc^{1+m/2} \cdot c^{-n_0/3} \cdot c^{5n/6} \le c^n.$$

Thus $|a_n|_{\mathsf{v}} \le c^n$, as claimed.

To show (2), we now assume in addition that all roots of $p_0(z)$ are contained in $\{0\} \cup \mathcal{U}$. Let $\mathsf{v} \in S$. We show that there exists $c \in \mathbb{R}_{\ge 1}$ such that $|a_n|_{\mathsf{v}} \le n^{c \log n}$ for all sufficiently large n. To this end, first note that we may choose $c_0 \in \mathbb{R}_{\ge 1}$ and enlarge $n_0 > m$ so that $dc^3 n^{s+1} \le n^{c \log(n)}$ for all $n \ge n_0$ and $c \ge c_0$. Now let $c \in \mathbb{R}_{\ge c_0}$ be sufficiently large so that $|a_n|_{\mathsf{v}} \le c$ for all $n \in \{1, \dots, n_0\}$ and $\|B_i\|_{\mathsf{v}} \le c$ for all $i \in \{1, \dots, d\}$. By our assumption on the roots of $p_0(z)$, all the eigenvalues of A are contained in $\{0\} \cup \mathcal{U}$. Thus $\|A^n\|_{\mathsf{v}} \in O(n^s)$ by Lemma 5.1, and we can also assume $\|A^n\|_{\mathsf{v}} \le cn^s$ for $n \ge 1$, enlarging c if necessary.

Since $k \ge 2 > 1 + m/n_0$, enlarging c further, we may also assume

$$c(\log k - \log(1 + m/n_0)) > s + 1$$

and even $dc^2 \le n_0^{c(\log k - \log(1+m/n_0)) - (s+1)}$. Then $dc^2 \le n^{c(\log k - \log(1+m/n)) - (s+1)}$ for all $n \ge n_0$.

We show $\|\mathbf{a}_0(n)\|_{\mathsf{v}} \le n^{c \log n}$ for all $n \ge n_0$ by induction. Let $n \ge n_0$. Equation (6) gives

$$\|\mathbf{a}_0(n)\|_{\mathsf{v}} \le \varepsilon_{\mathsf{v}}(dn) cn^s \cdot \max\Big\{\|\mathbf{a}_0(n_0)\|_{\mathsf{v}},\ \|B_i\|_{\mathsf{v}} \cdot \|\mathbf{a}_i(n+m-j)\|_{\mathsf{v}}\Big\},$$

where $i \in \{1, \dots, d\}$ and $j \in \{0, \dots, n-n_0-1\}$. By induction hypothesis we can estimate

$$\|\mathbf{a}_i(n+m-j)\|_{\mathsf{v}} \ \le\ \max\Big\{c, \Big(\frac{n+m}{k}\Big)^{c \log\left(\frac{n+m}{k}\right)}\Big\}.$$

With the latter bound of the maximum,

$$\begin{aligned}\|\mathbf{a}_0(n)\|_{\mathsf{v}} &\le dcn^{s+1} \cdot c \cdot \Big(\frac{n+m}{k}\Big)^{c \log\left(\frac{n+m}{k}\right)} \le dc^2 n^{s+1+c\log\left(\frac{n+m}{k}\right)} \\ &\le dc^2 n^{s+1+c\log(n)+c\log\left(1+\frac{m}{n}\right)-c\log(k)} \le n^{c\log(n)};\end{aligned}$$

in case $\|\mathbf{a}_i(n+m-j)\|_{\mathsf{v}} \le c$ we have

$$\|\mathbf{a}_0(n)\|_{\mathsf{v}} \le dc^3 n^{s+1} \le n^{c \log(n)}. \qquad \square$$

We have thus established the general growth bound for the coefficients of a Mahler function: the height of the nth coefficient is at most linear in n.

6. First gap: characterization of totally analytic Mahler functions

In this section, we characterize k-Mahler functions $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ with $h(a_n) \in o(n)$. Let $f(z) \in \overline{\mathbb{Q}}[\![z]\!]$ be an M-function. There exists a number field K such that $f(z) \in K[\![z]\!]$, and for every place v of K, we may consider $f(z)$ as a power series over the algebraic closure C_{v} of the completion K_{v}. Then $f(z)$ has a positive radius of convergence, and it is meromorphic in the open unit disk of C_{v}. Moreover, for all but finitely many places of K, the radius of convergence of $f(z)$ is equal to 1. Hence, an

M-function is *globally analytic*. We say that $f(z)$ is *totally analytic* if, for every place v of K, $f(z)$ is analytic in the open unit disk of C_{v}.

Theorem 6.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be a k-Mahler function. The following statements are equivalent.*

(a) *We have $h(a_n) \in o(n)$.*
(b) *Every non-zero root of the k-Mahler denominator of $f(z)$ belongs to $\mathcal{U}$ (i.e., is a root of unity).*
(c) *The power series $f(z)$ is totally analytic.*
(d) *We have $h(a_n) \in O(\log^2 n)$.*

The crucial step here lies in showing that all roots of the k-Mahler denominator are contained in $\{0\} \cup \mathcal{U}$. This relies on the deep results on Mahler's method by Nishioka, Philippon, Fernandes, as well as by Adamczewski and Faverjon that were recalled in Section 4. But first we need the following easy lemma.

Lemma 6.2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be a power series that is* not *a polynomial. If $h(a_n) \in o(n)$, then $f(z)$ has radius of convergence 1 for every absolute value of $\overline{\mathbb{Q}}$.*

Proof. Fix an absolute value $|\cdot|$ on $\overline{\mathbb{Q}}$, and let

$$\rho = \Big(\limsup_{n\to\infty} \sqrt[n]{|a_n|} \Big)^{-1} \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

be the radius of convergence of $f(z)$.

We show $\rho = 1$ by contradiction. Suppose first $\rho > 1$. Choose $\rho' \in \mathbb{R}_{>0}$ with with $1 < \rho' < \rho$. Since $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1/\rho < 1/\rho'$, we have $|a_n| \leq (1/\rho')^n$ for all sufficiently large n. Since f is not a polynomial, there exist infinitely many such n with $a_n \neq 0$, and for these $|a_n^{-1}| \geq (\rho')^n$. It follows that $h(a_n) \geq \log|a_n^{-1}| \geq n\log(\rho')$, in contradiction to our assumption.

Suppose now $\rho < 1$, and choose $\rho < \rho' < 1$. Then, for all $n_0 \geq 0$, there exists an $n \geq n_0$ such that $|a_n| \geq (1/\rho')^n$. Hence $h(a_n) \geq \log|a_n| \geq n\log(1/\rho')$ again yields a contradiction. □

We also require the notion of Cartier operators in the next proof.

Definition 6.3. For every $r \in \Sigma_k$, we define a *Cartier operator* $\Delta_r \colon \overline{\mathbb{Q}}[\![z]\!] \to \overline{\mathbb{Q}}[\![z]\!]$ by

$$\Delta_r\Big(\sum_{n=0}^{\infty} a_n z^n \Big) = \sum_{n=0}^{\infty} a_{kn+r} z^n \,.$$

Note that, if $p(z) \in \overline{\mathbb{Q}}[z]$, then $\deg(\Delta_r(p(z))) \leq (\deg p)/k$. Moreover, if $j \geq 1$, then a short computation yields

$$\Delta_r\Big(p(z) f(z^{k^j})\Big) = \Delta_r(p(z)) \cdot f(z^{k^{j-1}}).$$

Proposition 6.4. *Let $f(z) \in \overline{\mathbb{Q}}[\![z]\!]$ be k-Mahler and let $\mathfrak{d}(z) \in \overline{\mathbb{Q}}[z]$ be its k-Mahler denominator. If $\lambda \in \overline{\mathbb{Q}}$ is a root of $\mathfrak{d}(z)$, and $|\cdot|$ is an absolute value on $\overline{\mathbb{Q}}$ with $0 < |\lambda| < 1$, then the radius of convergence of $f(z)$ with respect to this absolute value is strictly less than 1.*

Proof. Let us first consider a minimal homogeneous equation associated with $f(z)$:

$$p_0(z)f(z) + p_1(z)f(z^k) + \cdots + p_d(z)f(z^{k^d}) = 0\,. \tag{7}$$

By minimal, we mean that $p_0(z)p_d(z) \neq 0$, that d is minimal, and that $p_0(z), \ldots, p_d(z) \in \overline{\mathbb{Q}}[z]$ are relatively prime. If $f(z) = 0$, then we can take $d = 0$ and $p_0(z) = 1$. Thus also $\mathfrak{d}(z) = 1$ and the claim is trivially true. We may assume $f(z) \neq 0$, so that $d \geq 1$.

As in Section 4, we can assume that there exists a number field K containing λ and all coefficients of $f(z)$ as well as the coefficients of the polynomials $p_0(z), \ldots, p_d(z)$. Further, $|\cdot|$ on K arises from a place v of K, and C_v is the algebraic closure of the completion K_v, with $\overline{K}$ denoting the algebraic closure of K inside C_v.

We claim that $f(z), \ldots, f(z^{k^{d-1}})$ are linearly independent over $\overline{K}(z)$. By way of contradiction, suppose that this is not the case. Then there exist $q_0(z), \ldots, q_{d-1}(z) \in \overline{K}[z]$, not all zero, such that

$$q_0(z)f(z) + \cdots + q_{d-1}(z)f(z^{k^{d-1}}) = 0.$$

Since the coefficients of $q_0(z), \ldots, q_d(z)$ are all algebraic over $\mathbb{Q}$, we can assume $q_0(z), \ldots, q_d(z) \in \overline{\mathbb{Q}}[z]$. Let $t \in \{0, \ldots, d-1\}$ be minimal with $q_t(z) \neq 0$, say $q_t(z) = \sum_{j=m}^{M} b_j z^j$ with $b_m, b_M \neq 0$. Let $m = \sum_{\nu=0}^{\infty} k^\nu m_\nu$ with $m_\nu \in \Sigma_k$ be the base-k expansion of m (with all but finitely many m_ν being zero). Setting $\Delta = \Delta_{m_t} \circ \cdots \circ \Delta_{m_0}$, we see $\Delta(q_t(z)) \neq 0$. Then

$$\Delta(q_t(z))f(z) + \cdots + \Delta(q_{d-1}(z))f(z^{k^{d-1-t}}) = 0$$

is a k-Mahler equation for $f(z)$, contradicting the minimality of d. Therefore $f(z), \ldots, f(z^{k^{d-1}})$ must be linearly independent over $\overline{K}(z)$, as claimed.

Now

$$\begin{pmatrix} f(z) \\ \vdots \\ f(z^{k^{d-1}}) \end{pmatrix} = A(z) \begin{pmatrix} f(z^k) \\ \vdots \\ f(z^{k^d}) \end{pmatrix}$$

with

$$A(z) = \begin{pmatrix} -\frac{p_1(z)}{p_0(z)} & -\frac{p_2(z)}{p_0(z)} & \cdots & -\frac{p_{d-1}(z)}{p_0(z)} & -\frac{p_d(z)}{p_0(z)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathrm{GL}_d(K(z)).$$

By Corollary 4.4, we have that

$$f(\lambda^{k^n}), f(\lambda^{k^{n+1}}), \ldots, f(\lambda^{k^{n+d-1}})$$

are linearly independent over $\overline{K}$, as soon as n is large enough, say $n \geq n_0$.

Now, iterating Equation (7), we obtain an equation of the form

$$r_0(z)f(z) + r_1(z)f(z^{k^{n_0}}) + \cdots + r_d(z)f(z^{k^{n_0+d-1}}) = 0\,, \tag{8}$$

where we assume without any loss of generality that $r_0(z), \ldots, r_d(z) \in K[z]$ are relatively prime. We claim that $f(z)$ has a pole at λ. Let us assume by contradiction that $f(z)$ is well-defined at λ. Since $\mathfrak{d}(\lambda)=0$, it follows that $r_0(\lambda)=0$ and we get that

$$r_1(\lambda)f(\lambda^{k^{n_0}})+\cdots+r_d(\lambda)f(\lambda^{k^{n_0+d-1}})=0\,.$$

Since $f(\lambda^{k^{n_0}})$, …, $f(\lambda^{k^{n_0+d-1}})$ are linearly independent over $\overline{K}$, all the $r_i(z)$ should vanish at λ, contradicting the fact that they are relatively prime. Hence, $f(z)$ has a pole at λ and its radius of convergence is therefore less than 1. ☐

We now have the ingredients to characterize Mahler functions with $h(a_n)\in o(n)$.

Proof of Theorem 6.1. Let $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\overline{\mathbb{Q}}[\![z]\!]$ be k-Mahler.

(a) ⇒ (c) Suppose $h(a_n)\in o(n)$. By Lemma 6.2 the series $f(z)$ has radius of convergence at least 1 with respect to every absolute value $|\cdot|$ on $\overline{\mathbb{Q}}$.

(c) ⇒ (b) Suppose now $f(z)$ has radius of convergence at least 1 with respect to every absolute value $|\cdot|$ on $\overline{\mathbb{Q}}$. Let $\mathfrak{d}\in\overline{\mathbb{Q}}[z]$ be the k-Mahler denominator of $f(z)$. Suppose there exists $0\neq\lambda\in\overline{\mathbb{Q}}$ with $\mathfrak{d}(\lambda)=0$ such that λ is *not* a root of unity. By Kronecker's Theorem there exists an absolute value $|\cdot|$ on $\overline{\mathbb{Q}}$ for which $|\lambda|<1$. By Proposition 6.4, the series $f(z)$ has radius of convergence strictly less than 1 for this absolute value, a contradiction.

(b) ⇒ (d) Suppose all roots of the k-Mahler denominator $\mathfrak{d}(z)\in\overline{\mathbb{Q}}[z]$ of $f(z)$ are contained in $\{0\}\cup\mathcal{U}$. Then $h(a_n)\in O(\log^2 n)$ by (2) of Proposition 5.2.

(d) ⇒ (a) Clearly $h(a_n)\in O(\log^2 n)$ implies $h(a_n)\in o(n)$. ☐

7. Second gap: characterization of regular Mahler functions

In this section, we characterize Mahler functions $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\overline{\mathbb{Q}}[\![z]\!]$ with $h(a_n)\in o(\log^2 n)$. The following result also proves Case (b) of Theorem 1.2.

Theorem 7.1. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\overline{\mathbb{Q}}[\![z]\!]$ be a k-Mahler function. The following statements are equivalent.*

(a) *We have $h(a_n)\in o(\log^2 n)$.*
(b) *Every non-zero root of the k-Mahler denominator of $f(z)$ belongs to $\mathcal{U}_k$.*
(c) *The power series $f(z)$ is k-regular.*
(d) *We have $h(a_n)\in O(\log n)$.*

We already know that, if $h(a_n)\in o(\log^2 n)$, then every root ζ of the k-Mahler denominator $\mathfrak{d}(z)$ of $f(z)$ is contained in $\{0\}\cup\mathcal{U}$. The brunt of the work in this section lies in showing $\zeta\in\{0\}\cup\mathcal{U}_k$, that is, if $\zeta\neq 0$, then $\zeta^{k^j}\neq\zeta$ for all $j>0$. This requires a careful analysis of the asymptotics of $f(z)$ at such a hypothetical root of $\mathfrak{d}(z)$ to establish a contradiction.

We start with some estimates.

Lemma 7.2. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathbb{R}[\![z]\!]$ be a power series with nonnegative coefficients. Suppose there exists $c\in\mathbb{R}_{>0}$ with $a_n\leq n^{c\log n}$ for all sufficiently large n. Let c', $\varepsilon\in\mathbb{R}_{>0}$ with $c'>2c$. Then there exists $t_0\in[0,1)$ such that*

$$\sum_{n=\lceil m\log^2 m\rceil}^{\infty}a_nt^n<\varepsilon\qquad\textit{for all } t\in[t_0,1)\textit{ and } m\geq\frac{c'}{1-t}\,.$$

Proof. By assumption, for large n,

$$a_n t^n \leq \exp(c \log^2 n + n \log t).$$

We will show that, for sufficiently large m (ensured by choice of t_0) and $n \geq m \log^2 m$,

$$c \log^2 n + n \log t \leq \tfrac{1}{2} n \log t. \tag{9}$$

We first show how to conclude the proof using Eq. (9). Then $a_n t^n \leq t^{n/2}$ and

$$\sum_{n=\lceil m \log^2 m \rceil}^{\infty} a_n t^n \leq \frac{t^{\lceil m \log^2 m \rceil/2}}{1-\sqrt{t}} = \frac{(1+\sqrt{t})t^{\lceil m \log^2 m \rceil/2}}{1-t} < \frac{2t^{\lceil m \log^2 m \rceil/2}}{1-t}.$$

We need to bound the right side by a constant. Using $m \geq c'/(1-t)$ and $t \in [0,1)$, we have

$$\log\left(\frac{2t^{\lceil m \log^2 m \rceil/2}}{1-t}\right) \leq \log 2 + \frac{c' \log t}{2(1-t)} \log^2\left(\frac{c'}{1-t}\right) - \log(1-t). \tag{10}$$

Recall $\lim_{t \to 1} \log t/(1-t) = -1$ and $\log^2(c'/(1-t)) \sim \log^2(1-t)$ for $t \to 1^-$. Hence the right side of Eq. (10) tends to $-\infty$ as $t \to 1^-$. Choosing $t_0 \in [0,1)$ sufficiently close to 1, therefore

$$\sum_{n=\lceil m \log^2 m \rceil}^{\infty} a_n t^n \leq \varepsilon \qquad \text{for } t \in [t_0, 1) \text{ and } m \geq c'/(1-t).$$

It remains to show the bound in Eq. (9). The latter is equivalent to $c \log^2 n + \frac{1}{2} n \log t \leq 0$. Since $\log t \leq t - 1 \leq -c'/m$, it suffices to show

$$c \log^2 n - n\frac{c'}{2m} \leq 0 \qquad \text{for } n \geq m \log^2 m. \tag{11}$$

We first show this for $n = m \log^2 m$. Now

$$c \log^2(m \log^2 m) - m(\log^2 m)\frac{c'}{2m} \sim (c - c'/2) \log^2 m$$

as a function in m for $m \to \infty$, and $c - c'/2$ is negative by choice of c'. Thus, for sufficiently large m, we have $c \log^2(m \log^2 m) - m(\log^2 m)\frac{c'}{2m} \leq 0$. We can ensure a large enough m by choosing $t_0 \in [0,1)$ sufficiently close to 1.

Now, set $g(n) := c \log^2 n$ and $h(n) := n\frac{c'}{2m}$. Then $g'(n) = \frac{2c \log n}{n}$, and hence

$$g'(m \log^2 m) = \frac{2c \log m + 2c \log(\log^2 m)}{m \log^2 m} \sim \frac{2c}{m \log m}.$$

Thus, choosing m sufficiently large, we may also ensure

$$g'(m \log^2 m) \leq h'(m \log^2 m) = \frac{c'}{2m}.$$

Since $g(n)$ is concave for $n \geq \exp(1)$, this ensures $g(n) \leq h(n)$ for $n \geq m \log^2 m$. This proves Eq. (9) and ends the proof of the lemma. $\square$

Lemma 7.3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{R}[\![z]\!]$ *be a power series with nonnegative coefficients. Let* $a \in \mathbb{R}_{\geq 0}$ *and* $b,\, c \in \mathbb{R}_{>0}$. *Assume that there exist a sequence* $(t_j)_{j\geq 0} \to 1$ *in* $[0,1)$ *and* $m_0 \in \mathbb{Z}_{\geq 0}$ *such that*

$$f(t_j) \geq (1-t_j)^a \exp(c\log^2 m) t_j^{mb} \qquad \text{for all } j \geq 0 \text{ and } m \geq m_0.$$

Then there exist $c' \in \mathbb{R}_{>0}$ *and infinitely many* $n \geq 1$ *with* $a_n > \exp(c'\log^2 n)$.

Proof. Without any loss of generality we can assume that there exists a constant $c' > 2c$ such that $a_n \leq \exp(c'\log(n)^2)$ for all sufficiently large n. Indeed, otherwise the result holds trivially. By the previous lemma, for all sufficiently large j and $m \geq 3c'/(1-t_j)$, we have

$$\sum_{n=\lceil m\log^2 m\rceil}^{\infty} a_n t_j^n \leq 1\,.$$

Let $m_j = \lceil 3c'/(1-t_j)\rceil$ and

$$\begin{aligned} A_j &= \log\left((1-t_j)^a \exp(c\log^2 m_j) t_j^{m_j b}\right) \\ &= a\log(1-t_j) + c\log^2 m_j + m_j b \log t_j\,. \end{aligned}$$

Then, for sufficiently large j,

$$\sum_{n=0}^{\lfloor m_j \log^2 m_j\rfloor} a_n t_j^n \geq \exp(A_j) - 1\,.$$

Since

$$\Big(\frac{3c'}{(1-t_j)} + 1\Big) b\log t_j \leq m_j b\log t_j \leq \frac{3c'}{(1-t_j)} b\log t_j\,,$$

and $((\log t_j)/(1-t_j))_{j\geq 0} \to -1$, we find $(m_j b\log t_j)_{j\geq 0} \to -3c'b$. Since $\log^2(1/(1-t_j)) = \log^2(1-t_j)$ and therefore $\log^2 m_j \sim \log^2(1-t_j)$, we see that $A_j \sim c\log^2 m_j$. Choosing j sufficiently large, we may assume

$$A_j \geq \frac{c}{2}\log^2 m_j\,.$$

Therefore, again restricting to large enough j for the last inequality,

$$\sum_{n=0}^{\lfloor m_j \log^2 m_j\rfloor} a_n t_j^n \geq \exp(A_j) - 1 \geq \exp\Big(\frac{c}{4}\log^2 m_j\Big)\,.$$

By the pigeonhole principle, there exists $0 \leq n_j \leq m_j \log^2 m_j$ such that

$$a_{n_j} \geq \exp\Big(\frac{c}{4}\log^2 m_j\Big)/(1 + m_j\log^2 m_j)\,.$$

Thus

$$\log a_{n_j} \geq \frac{c}{4}\log^2 m_j - \log(m_j\log^2 m_j + 1) \sim \frac{c}{4}\log^2 m_j\,.$$

We may assume $\log a_{n_j} \geq \frac{c}{8}\log^2 m_j$. To finish, since $n_j \leq m_j \log^2 m_j$, we have

$$\log^2 n_j \leq \log^2(m_j\log^2 m_j) \sim \log^2 m_j\,.$$

We may take $\log^2 n_j \leq 2\log^2 m_j$, so that $\log a_{n_j} \geq \frac{c}{16}\log^2 n_j$. Since $\log a_{n_j} \geq \frac{c}{8}\log^2 m_j$ and $(m_j)_{j\geq 0} \to \infty$, also $(n_j)_{j\geq 0} \to \infty$. Thus there are in fact infinitely many distinct such n_j. □

Lemma 7.4. *Let $\zeta \in \mathbb{C}$ with $\zeta^k = \zeta$. Let $p(z) \in \mathbb{C}[z]$ with $p(0) = 1$ and $p(\zeta) \neq 0$. Then there exists $c \in \mathbb{R}_{>0}$ such that, for all $t \in [0, 1)$ with $p(\zeta t^{k^n}) \neq 0$ for all $n \geq 0$,*

$$\left| \left(\prod_{n=0}^{\infty} p(\zeta t^{k^n}) \right)^{-1} \right| > |1 - t|^c .$$

Proof. The proof is the same as the one of the lower bound in [AB17, Lemma 9.5 and Proposition 9.2]. Let α_1, ..., α_s denote the roots of $p(z)$ (with multiplicity). Then

$$p(z) = (1 - \alpha_1^{-1}z) \cdots (1 - \alpha_s^{-1}z) .$$

It suffices to show the claim for $1 - \alpha_1^{-1}z$. Suppose $t \in [0, 1]$ is such that $\zeta t^{k^n} \neq \alpha_1$ for all $n \geq 0$. Then the infinite product $\prod_{n=0}^{\infty}(1 - \alpha_1^{-1}t^{k^n})^{-1}$ converges, and

$$\left| \prod_{n=0}^{\infty} \frac{1}{1 - \alpha_1^{-1}\zeta t^{k^n}} \right| \geq \prod_{n=0}^{\infty} \frac{1}{1 + |\alpha_1^{-1}| t^{k^n}} \geq \prod_{n=0}^{\infty} \exp(-|\alpha_1^{-1}| t^{k^n}) .$$

Then, by [AB17, Lemma 9.4],

$$\prod_{n=0}^{\infty} \exp(-|\alpha_1^{-1}| t^{k^n}) \geq \exp\Big(- |\alpha_1^{-1}|(1 - 1/k)^{-1} \sum_{n=1}^{\infty} \frac{t^n}{n} \Big) = (1 - t)^{\frac{|\alpha_1^{-1}|k}{k-1}} .$$ □

Let $B(\lambda, r) \subseteq \mathbb{C}$, respectively $\overline{B(\lambda, r)} \subseteq \mathbb{C}$, denote the open, respectively closed, disc of radius $r \in \mathbb{R}_{\geq 0}$ with center $\lambda \in \mathbb{C}$.

Lemma 7.5 (Special case of [AB17, Lemma 10.2]). *Let $d \in \mathbb{Z}_{>0}$, let $0 \neq \zeta \in \mathbb{C}$ such that $\zeta^k = \zeta$, and let $A \colon \overline{B(0, 1)} \to \mathbb{C}^{d\times d}$ be a continuous, matrix-valued function. Assume that $w(z) \in \mathbb{C}[\![z]\!]^d$ satisfies the equation*

$$w(\lambda) = A(\lambda)w(\lambda^k) \qquad \text{for all } \lambda \in B(0, 1) .$$

Assume also that the following properties hold.

(i) *The coordinates of $w(z)$ are analytic in $B(0, 1)$.*
(ii) *The matrix $A(\zeta)$ is* not *nilpotent.*
(iii) *The set $\{ w(\lambda) : \lambda \in B(0, 1) \}$ is not contained in a proper vector subspace of $\mathbb{C}^d$.*

Then there exist $c \in \mathbb{R}_{>0}$ and a sequence $(t_j)_{j\geq 0} \to 1$ in $[0, 1)$ such that

$$\|w(t_j\zeta)\| > |1 - t_j|^c \qquad \text{for all } j \geq 0 .$$

Proof. This is [AB17, Lemma 10.2] in the special case $\theta = 0$. We do not assume $w(z)$ to be continuous in $\overline{B(0, 1)}$, but this assumption is never used in the proof and is therefore superfluous. □

Lemma 7.6. *Let $b \in \mathbb{Z}_{>0}$ and a, $a' \in \mathbb{R}$ with $a' > a > 0$. Then there exists $t_0 \in [0, 1)$ such that*

$$(1 - t^{1/b})^a > (1 - t)^{a'} \qquad \text{for all } t \in [t_0, 1).$$

Proof. For $t \in [0,1)$ we have

$$1-t = (1-t^{1/b}) \sum_{i=0}^{b-1} t^{i/b} \le b(1-t^{1/b}).$$

Moreover $(1-t)^{a'-a} < 1/b^a$ for t sufficiently close to 1. Then

$$(1-t^{1/b})^a \ge \frac{(1-t)^{a'}}{(1-t)^{a'-a} b^a} > (1-t)^{a'}. \qquad \square$$

Armed with these estimates, we can finally prove a further restriction on the roots of the k-Mahler denominator. This is the key step in the current section. The arguments are in many aspects very similar to those used by Adamczewski and Bell in [AB17, §11].

Proposition 7.7. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[\![z]\!]$ be a k-Mahler series that is analytic in $B(0,1)$. Let $\zeta \in \mathcal{U}$ with $\zeta^{k^{j_0}} = \zeta$ for some $j_0 \ge 1$, and let $l = k^{j_0}$. Suppose there exists an l-Mahler equation*

$$p_0(z) f(z) = p_1(z) f(z^l) + \cdots + p_d(z) f(z^{l^d}),$$

with $p_0(z), \ldots, p_d(z) \in \mathbb{C}[z]$ coprime, with $p_0(z)p_d(z) \neq 0$, and such that $p_0(\zeta) = 0$.

Then there exist $a,\ b,\ c \in \mathbb{R}_{>0}$, $m_0,\ n_0 \in \mathbb{Z}_{\ge 0}$, and a sequence $(t_j)_{j\ge 0} \to 1$ in $[0,1)$ such that

$$\left| \sum_{n=0}^{\infty} a_{n+n_0} (\zeta t_j)^n \right| \ge (1-t_j)^a \exp(b \log^2 m) t_j^{mc} \qquad \text{for all } j \ge 0 \text{ and } m \ge m_0\,.$$

Proof. Applying Lemma 3.1, there exist $n_0 \ge 0$ and $q_0(z), \ldots, q_{d+1}(z) \in \mathbb{C}[z]$ such that $a_{n_0} \neq 0$ and

$$f_0(z) = \sum_{n=0}^{\infty} a_{n+n_0} z^n$$

satisfies

$$q_0(z) f_0(z) = q_1(z) f_0(z^l) + \cdots + q_{d+1} f_0(z^{l^{d+1}})\,,$$

with $q_0(0) = 1$, with $q_0(\zeta) = 0$, and with $q_i(\zeta) \neq 0$ for some $i \in \{1, \ldots, d+1\}$.

Let $\nu_i \in \mathbb{Z}_{\ge 0}$ be the order of vanishing of $q_i(z)$ at ζ. Define

$$r \coloneqq \min \left\{ \frac{\nu_i + (i-1)\nu_0}{i} : i \in 1, \ldots, d+1 \right\} \in \mathbb{Q}_{\ge 0}\,.$$

Since $\nu_0 > 0$ and $\nu_i = 0$ for some $i \in \{1, \ldots, d+1\}$, we have $r < \nu_0$. Defining

$$g(z) \coloneqq f_0(\zeta z) \prod_{n=0}^{\infty} \frac{q_0(\zeta z^{l^n})}{(1-z^{l^n})^r}$$

we obtain

$$g(z) = \sum_{i=1}^{d+1} r_i(z) g(z^{l^i}) \qquad \text{with} \qquad r_i(z) = q_i(\zeta z) \frac{1}{(1-z)^r} \prod_{n=1}^{i-1} \frac{q_0(\zeta z^{l^n})}{(1-z^{l^n})^r} \in \mathbb{C}(z)\,.$$

In the expression for $r_i(z)$, the denominator has roots at every $\omega \in \mathbb{C}$ for which $\omega^{l^{i-1}} = 1$. If $\omega \neq 1$, then $r < \nu_0$ guarantees that $r_i(z)$ does not actually have a pole at ω. For $\omega = 1$, this is ensured by $ir \leq \nu_i + (i-1)\nu_0$. Thus, all $r_i(z)$ are in fact polynomials. Moreover, by choice of r, there exists an $i_0 \in \{1, \dots, d+1\}$ such that $r_{i_0}(1) \neq 0$.

Claim: There exist $a \in \mathbb{R}_{>0}$ and a sequence $(t_j)_{j\geq 0} \to 1$ in $[0,1)$ with

$$|g(t_j)| \geq (1-t_j)^a.$$

Proof of Claim. First we deal with the degenerate case in which $g(z)$ is constant. Then $g(z) = g(0)$ and from the definition of g we see $g(0) \neq 0$ since $f_0(0) \neq 0$ and $q_0(0) = 1$. Choosing $a = 1$, any sequence $(t_j)_{j\geq 0} \to 1$ in $[0,1)$ satisfies $|g(t_j)| > 1 - t_j$ for sufficiently large j. From now on we may assume that $g(z)$ is *not* constant.

We are going to apply Lemma 7.5. Denote by $\|\cdot\|$ the maximum norm with respect to $|\cdot|$. Let $w(z) = \Big(g(z), g(z^l), \dots, g(z^{l^d})\Big)^T$ and

$$A(z) = \begin{pmatrix} r_1(z) & r_2(z) & \dots & r_{d-1}(z) & r_d(z) & r_{d+1}(z) \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{C}(z)^{(d+1)\times(d+1)}.$$

Then $w(z) = A(z)w(z^l)$. The coordinates of $A(z)$ are polynomials and hence of course continuous. We verify the conditions of Lemma 7.5.

(i) The coordinates of $w(z)$ are analytic in $B(0,1)$ since $g(z)$ is analytic in $B(0,1)$.

(ii) The characteristic polynomial of $A(z)$ is $y^{d+1} - r_1(z)y^d - \cdots - r_{d+1}(z) \in \mathbb{C}(z)[y]$. Since $r_{i_0}(1) \neq 0$, the matrix $A(1)$ is not nilpotent.

(iii) Suppose that $S = \{\, w(\lambda) : \lambda \in B(0,1) \,\}$ is contained in a proper subspace of $\mathbb{C}^d$. Then there exist $\alpha_0, \dots, \alpha_d \in \mathbb{C}$, not all zero, such that $\alpha_0 g(\lambda) + \cdots + \alpha_d g(\lambda^{l^d}) = 0$ for all $\lambda \in B(0,1)$. Since $g(z)$ is analytic in $B(0,1)$ this forces $\alpha_0 g(z) + \cdots + \alpha_d g(z^{l^d}) = 0$. But then $g(z)$ is constant by [AB17, Lemma 7.9], a contradiction. Hence the set S is not contained in a proper subspace of $\mathbb{C}^d$.

Applying Lemma 7.5, there exist $a \in \mathbb{R}_{>0}$ and a sequence $(t_j)_{j\geq 0} \to 1$ in $[0,1)$ such that

$$\|w(t_j)\| > (1-t_j)^a \qquad \text{for all } j \geq 0\,.$$

Restricting to a subsequence and making a substitution, we may assume that there exists $i \in [0,d]$ and $b = l^i$ such that $|g(t_j)| > (1 - t_j^{1/b})^a$ for $j \geq 0$. Applying Lemma 7.6 and replacing a by a slightly larger constant, we may actually take $|g(t_j)| > (1-t_j)^a$ for all $j \geq 0$. ☐ (Claim)

By the result of Mahler, see Example (d) in Section 2, there exists a constant $c \in \mathbb{R}_{>0}$ such that, for some $m_0 \geq 0$,

$$\prod_{n=0}^{\infty} (1 - t_j^{l^n})^{-1} \geq \sum_{n=m_0}^{\infty} \exp(c \log^2 n) t_j^n\,.$$

Thus

$$\prod_{n=0}^{\infty} (1 - t_j^{l^n})^{-1} \geq \exp(c \log^2 m) t_j^m \qquad \text{for all } m \geq m_0\,.$$

The lower bound for $g(z)$ together with the fact that $f_0(z)$ is analytic in $B(0,1)$ implies $q_0(\zeta t_j^{l^n}) \neq 0$ for all $n \geq 0$. By Lemma 7.4, there exists $a' \in \mathbb{R}_{>0}$ such that

$$\left| \prod_{n=0}^{\infty} \frac{(1-t_j^{l^n})^{\nu_0}}{q_0(\zeta t_j^{l^n})} \right| > (1-t_j)^{a'} \qquad \text{for } j \text{ large enough.}$$

With $b = \nu_0 - r > 0$, we conclude, for $m \geq m_0$, that

$$|f_0(t_j\zeta)| = |g(t_j)| \cdot \left| \prod_{n=0}^{\infty} \frac{(1-t_j^{l^n})^{r}}{q_0(\zeta t_j^{l^n})} \right| > (1-t_j)^{a+a'} \exp(cb \log^2 m) t_j^{mb}. \qquad \square$$

Proposition 7.8. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ be k-Mahler and suppose that $h(a_n) \in o(\log^2 n)$. Then the roots of the k-Mahler denominator of $f(z)$ are contained in $\{0\} \cup \mathcal{U}_k$.*

Proof. Let $\mathfrak{d}(z)$ be the k-Mahler denominator of $f(z)$. Suppose that $0 \neq \zeta \in \overline{\mathbb{Q}}$ is such that $\mathfrak{d}(\zeta) = 0$. Then $\zeta \in \mathcal{U}$ by Theorem 6.1. We have to show $\zeta^{k^j} \neq \zeta$ for all $j \geq 1$.

Suppose to the contrary that $\zeta^{k^{j_0}} = \zeta$ for some $j_0 \geq 1$. Let $l = k^{j_0}$. Then $f(z)$ is also l-Mahler by Lemma 3.2. Let $p_0(z), \dots, p_d(z) \in \overline{\mathbb{Q}}[z]$ be coprime, with $p_0(z)p_d(z) \neq 0$, such that

$$p_0(z)f(z) = p_1(z)f(z^l) + \cdots + p_d(z)f(z^{l^d}).$$

Since this is also a k-Mahler equation for $f(z)$, the k-Mahler denominator $\mathfrak{d}(z)$ divides $p_0(z)$, hence $p_0(\zeta) = 0$. Fix any embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and thereby an archimedean absolute value on $\overline{\mathbb{Q}}$. We apply Proposition 7.7 to conclude that there exist $a, b, c \in \mathbb{R}_{>0}$, $m_0, n_0 \in \mathbb{Z}_{\geq 0}$, and a sequence $(t_j)_{j \geq 0} \to 1$ in $[0,1)$ such that

$$|f_0(\zeta t_j)| \geq (1-t_j)^a \exp(b \log^2 m) t_j^{mc} \qquad \text{for all } j \geq 0 \text{ and } m \geq m_0,$$

where $f_0(z) = \sum_{n=0}^{\infty} a_{n+n_0} z^n$. Since $\sum_{n=0}^{\infty} |a_{n+n_0}| t^n \geq |f_0(\zeta t)|$ for $t \in [0,1)$, the conditions of Lemma 7.3 are satisfied for $\sum_{n=0}^{\infty} |a_{n+n_0}| z^n$. Thus there exist $c' \in \mathbb{R}_{>0}$ such that $|a_n| \geq \exp(c' \log^2 n)$ infinitely often. Thus $h(a_n) \notin o(\log^2 n)$; a contradiction. $\square$

Once we know that all roots of the k-Mahler denominator of $f(z)$ are contained in $\{0\} \cup \mathcal{U}_k$ it is not hard to show that $f(z)$ is k-regular. This was shown by Dumas [Dum93, Théorème 30]; see also [BCCD19, Proposition 2]. We recall the proof.

Keep in mind that $\mathcal{U}_k$ consists of all roots of unity ζ for which $\zeta^{k^j} \neq \zeta$ for all $j \geq 1$. In particular, $1 \notin \mathcal{U}_k$. By [AB17, Proposition 7.8], the infinite product

$$\prod_{n=0}^{\infty} (1 - \zeta^{-1} z^{k^n})^{-1}$$

is k-regular for $\zeta \in \mathcal{U}_k$.

Proposition 7.9. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ be a k-Mahler series and suppose that $h(a_n) \in o(\log^2 n)$. Then $f(z)$ is a k-regular power series.*

Proof. Since every polynomial is k-regular, and sums of k-regular sequences are k-regular, it suffices to show the claim for $\sum_{n=0}^{\infty} a_{n+n_0} z^n$ for some $n_0 \geq 0$. By Lemma 3.1 we may therefore assume $\mathfrak{d}(0) = 1$ for the k-Mahler denominator $\mathfrak{d}(z)$ of $f(z)$. By Proposition 7.8, all roots of $\mathfrak{d}(z)$ are contained in $\mathcal{U}_k$. By Theorem 3.8 we can write

$$f(z) = \frac{g(z)}{\prod_{n=0}^{\infty} \mathfrak{d}(z^{k^n})}$$

with a k-Becker series $g(z)$. By a theorem of Becker [Bec94, Theorem 2], the series $g(z)$ is k-regular. Because

$$\prod_{n=0}^{\infty}(1-\zeta^{-1}z^{k^n})^{-1}$$

is k-regular for $\zeta \in \mathcal{U}_k$, and products of k-regular series are k-regular, also $f(z)$ is k-regular. □

Allouche and Shallit [AS92, Theorem 2.10] show $|a_n| = O(n^c)$ for a $\mathbb{C}$-valued k-regular sequence. A similar argument bounds the height of the coefficients.

Lemma 7.10. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ is k-regular, then $h(a_n) \in O(\log n)$.*

Proof. For $n \in \mathbb{Z}_{\geq 0}$, we recall that $\langle n \rangle_k \in \Sigma_k^*$ is the canonical base-k expansion of n. By Theorem 3.5 there exists a linear representation (u, μ, v) (of some dimension $d \in \mathbb{Z}_{\geq 0}$) such that $a_n = u\mu(\langle n \rangle_k)v$ for all $n \in \mathbb{Z}_{\geq 0}$. Moreover, using basic properties of the logarithmic Weil height (see [Wal00, Chapter 3]), we deduce that $h(u\mu(w)v) \in O(|w|)$ for $w \in \Sigma_k^*$. Noting that $|\langle n \rangle_k| \in O(\log n)$, we obtain $h(a_n) \in O(\log n)$. □

At this point, we are ready to prove Theorem 7.1.

Proof of Theorem 7.1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be a k-Mahler function.

(a) ⇒ (b) Suppose $h(a_n) \in o(\log^2 n)$. By Proposition 7.8 all roots of the k-Mahler denominator of $f(z)$ are contained in $\{0\} \cup \mathcal{U}_k$.

(b) ⇒ (c) Suppose that all roots of the k-Mahler denominator are contained in $\{0\} \cup \mathcal{U}_k$. Then $f(z)$ is k-regular by Proposition 7.9.

(c) ⇒ (d) Suppose $f(z)$ is k-regular. Then $h(a_n) \in O(\log n)$ by Lemma 7.10.

(d) ⇒ (a) Clearly $h(a_n) \in O(\log n)$ implies $h(a_n) \in o(\log^2 n)$. □

8. Third gap: word-convolution products of automatic sequences

In this section, we characterize Mahler functions $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ with $h(a_n) \in o(\log n)$. The arguments are similar to the ones used in [Bel05] and [BCH16]. As before, we actually prove a more extensive characterization involving a structural property.

Before stating the main result of this section, we first recall the definition of the word-convolution product following [BCH16].

Definition 8.1. Given two sequences of complex numbers $(a(n))_{n\geq 0}$ and $(b(n))_{n\geq 0}$, their *word-convolution product* is the sequence $a \star_w b$ defined by

$$a \star_w b(n) = \sum_{j=0}^{s} a([i_1 \cdots i_j]_k) b([i_{j+1} \cdots i_s]_k)\,,$$

where $\langle n \rangle_k = i_1 i_2 \cdots i_s \in \Sigma_k^*$.

We also need the notion of tame semi-group of matrices.

Definition 8.2. Let d be a positive integer. A semigroup of matrices $\mathcal{S} \subseteq K^{d\times d}$ is *tame*, if all eigenvalues of all matrices $A \in \mathcal{S}$ are contained in $\{0\} \cup \mathcal{U}$.

We are now ready to state the main result of this section. We already know that a k-Mahler function with $h(a_n) \in O(\log n)$ is k-regular. The sequence of coefficients $(a_n)_{n\geq 0}$ therefore has a minimal linear representation (u, μ, v) by Theorem 3.5.

Theorem 8.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be a k-Mahler function. The following statements are equivalent.*

(a) *We have $h(a_n) \in o(\log n)$.*
(b) *For every minimal linear representation (u, μ, v) of $(a_n)_{n\geq 0}$, the matrix semigroup $\mu(\Sigma_k^*)$ is tame.*
(c) *The sequence $(a_n)_{n\geq 0}$ is a $\overline{\mathbb{Q}}$-linear combination of word-convolution products of k-automatic sequences.*
(d) *We have $h(a_n) \in O(\log\log n)$.*

Our first goal is to use the additional restriction $h(a_n) \in o(\log n)$ to obtain a restriction on the possible eigenvalues of the matrices $\mu(w)$. The following lemma is similar to [Bel05, Lemma 2.3].

Lemma 8.4. *Let $(a_n)_{n\geq 0}$ be a k-regular sequence in $\overline{\mathbb{Q}}$, with a minimal linear representation (u, μ, v). Suppose $h(a_n) \in o(\log n)$. Then the finitely generated matrix semigroup $\mu(\Sigma_k^*)$ is tame.*

Proof. By definition of the linear representation, we have $a_{[w]_k} = u\mu(w)v$ for all $w \in \Sigma_k^*$. Since $[w]_k \in O(k^{|w|})$ for all $w \in \Sigma_k^*$, our assumption on the sequence translates into $h(u\mu(w)v) \in o(|w|)$.

Write d for the dimension of (u, μ, v). If $d = 0$, the claim is trivially true. Let $d > 0$. Suppose there exist a word $w \in \Sigma_k^*$ and $\lambda \in \overline{\mathbb{Q}} \setminus \{0\}$ not a root of unity such that λ is an eigenvalue of $\mu(w)$. Then there exists a nonzero vector $v_0 \in \overline{\mathbb{Q}}^{d\times 1}$ with $\mu(w)v_0 = \lambda v_0$. By minimality of the linear representation, there exist $w_1, \ldots, w_d \in \Sigma_k^*$ such that $\mu(w_1)v, \ldots, \mu(w_d)v$ form a basis of $\overline{\mathbb{Q}}^{d\times 1}$. Let $\alpha_1, \ldots, \alpha_d \in \overline{\mathbb{Q}}$ be such that $v_0 = \alpha_1\mu(w_1)v + \cdots + \alpha_d\mu(w_d)v$. Again by minimality, the set $\{\, u\mu(w') : w' \in \Sigma_k^* \,\}$ spans $\overline{\mathbb{Q}}^{1\times d}$. Therefore there exists $w' \in \Sigma_k^*$ such that $u\mu(w')v_0 \neq 0$.

Now
$$\sum_{i=1}^{d} \alpha_i u\mu(w'w^n w_i)v = u\mu(w')\mu(w^n)v_0 = \lambda^n u\mu(w')v_0 \neq 0.$$
Since λ is not a root of unity, there exists an absolute value $|\cdot|$ on $\overline{\mathbb{Q}}$ with $|\lambda| > 1$. We conclude that there exists an $i \in \{1, \ldots d\}$ with
$$|\alpha_i u\mu(w'w^n w_i)v| \geq |\lambda|^n \cdot \frac{|u\mu(w')v_0|}{d}$$
for infinitely many $n \geq 0$. Hence there exists $c' \in \mathbb{R}_{>0}$ such that, for these $n \geq 0$,
$$h(u\mu(w'w^n w_i)v) \geq \log|u\mu(w'w^n w_i)v| \geq nc'.$$
This is a contradiction. □

Tame semigroups afford a particular block diagonal decomposition.

Lemma 8.5. *Let $\mathcal{S} \subseteq \overline{\mathbb{Q}}^{d\times d}$ be a finitely generated tame semigroup. Then there exist $d_1, \ldots, d_r \in \mathbb{Z}_{\geq 0}$ with $d = d_1 + \cdots + d_r$, finite semigroups $\mathcal{S}_i \in \overline{\mathbb{Q}}^{d_i\times d_i}$ for $i \in \{1, \ldots, r\}$, and a matrix $T \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ such that*

$$T^{-1}\mathcal{S}T \subseteq \begin{pmatrix} \mathcal{S}_1 & \overline{\mathbb{Q}}^{d_1\times d_2} & \overline{\mathbb{Q}}^{d_1\times d_3} & \ldots & \overline{\mathbb{Q}}^{d_1\times d_r} \\ 0 & \mathcal{S}_2 & \overline{\mathbb{Q}}^{d_2\times d_3} & \ldots & \overline{\mathbb{Q}}^{d_2\times d_r} \\ 0 & 0 & \mathcal{S}_3 & \ldots & \overline{\mathbb{Q}}^{d_3\times d_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \mathcal{S}_r \end{pmatrix}.$$

Proof. If $\mathcal{S}$ spans $\overline{\mathbb{Q}}^{d\times d}$, then $\mu(\Sigma_k^*)$ is finite [BCH16, Lemma 4] and we are done. Otherwise, we iterate Lemma 5 of [BCH16] to get a block-upper-triangular decomposition with finite semigroup diagonals. □

The arguments in the following proof are similar to [Bel05, Theorem 2.6].

Proof of Theorem 8.3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be k-Mahler.

(a) ⇒ (b) Suppose $h(a_n) \in o(\log n)$. Then $(a_n)_{n\geq 0}$ is k-regular by Theorem 7.1. Lemma 8.4 implies that $\mu(\Sigma_k^*)$ is tame.

(b) ⇒ (d) Let (u, μ, v) be a minimal linear representation of the k-regular sequence $(a_n)_{n\geq 0}$. Suppose $\mu(\Sigma_k^*)$ is tame. We have to show $h(a_n) \in O(\log\log n)$. For this, it suffices to show $h(u\mu(w)v) \in O(\log|w|)$ for nonempty words $w \in \Sigma_k^*$. We can apply Lemma 8.5. Thus, there exists a finite semigroup $\mathcal{S}$ of block-diagonal matrices such that, after a change of basis, for every $w \in \Sigma_k^*$ the matrix $\mu(w)$ is of the form $D + N$ with $D \in \mathcal{S}$ and N strictly upper triangular. We may assume that $\mathcal{S}$ contains the identity matrix. Since Σ_k is finite, there exists a finite set $\mathcal{N}$ of strictly upper triangular matrices such that $\mu(\Sigma_k) \subseteq \mathcal{S} + \mathcal{N}$.

Let $w = a_1 \cdots a_l \in \Sigma_k^*$ with $a_1, \ldots, a_l \in \Sigma_k$, and let $\mu(a_i) = D_i + N_i$ with $D_i \in \mathcal{S}$ and $N_i \in \mathcal{N}$. For $J \subseteq \{1, \ldots, l\}$ with $J = \{j_1 < j_2 < \cdots < j_r\}$ define

$$b_J = uD_1 \cdots D_{j_1-1}N_{j_1}D_{j_1+1} \cdots D_{j_2-1}N_{j_2}D_{j_2+1} \cdots D_{j_r-1}N_{j_r}D_{j_r+1} \cdots D_l v\,.$$

Then

$$u\mu(w)v = u\mu(a_1 \cdots a_l)v = u(D_1 + N_1) \cdots (D_l + N_l)v = \sum_{J\subseteq\{1,\ldots,l\}} b_J\,.$$

Any product that includes d or more of the N_i's is 0, and hence $b_J = 0$ whenever $\#J \geq d$. Thus, the previous sum reduces to

$$u\mu(w)v = \sum_{\substack{J\subseteq\{1,\ldots,l\}\\ \#J<d}} b_J\,.$$

This sum has at most $\binom{l}{d-1} + \cdots + \binom{l}{0} \leq Cl^{d-1}$ nonzero terms for some constant $C \in \mathbb{R}_{>1}$. As $\mathcal{S}$ is a semigroup, each product $D_{j_i+1} \cdots D_{j_{i+1}-1}$ is again contained in the finite set $\mathcal{S}$. Hence

$$\#\{\, b_J : J \subseteq \{1, \ldots, l\} \,\} \leq d\,\#\mathcal{S}^d\,\#\mathcal{N}^{d-1} < \infty\,.$$

Let K be the number field generated by the finitely many coordinates of u, v, and $\mu(a)$ for $a \in \Sigma_k$. Then $b_J \in K$ for each $J \subseteq \{1, \ldots, l\}$. Since there are only finitely many of these elements, for every place v of K, there exists a constant $c_{\mathsf{v}} \in \mathbb{R}_{>0}$ such that $|b_J|_{\mathsf{v}} \le c_{\mathsf{v}}$ for all $J \subseteq \{1, \dots, l\}$, and we can take $c_{\mathsf{v}} = 1$ for all but finitely many places. For $m \in \mathbb{Z}_{\ge 0}$, let

$$\varepsilon_{\mathsf{v}}(m) = \begin{cases} m & \text{if } \mathsf{v} \text{ is archimedean,} \\ 1 & \text{if } \mathsf{v} \text{ is non-archimedean.} \end{cases}$$

Note $\prod_{\mathsf{v} \in M_K} \varepsilon_{\mathsf{v}}(m) \le m^{[K:\mathbb{Q}]}$. With this definition $|u\mu(w)v|_{\mathsf{v}} \le \varepsilon_{\mathsf{v}}(Cl^{d-1})c_{\mathsf{v}}$ and

$$\begin{aligned} h(u\mu(w)v) &= \log \prod_{\mathsf{v} \in M_K} \max\{1, |u\mu(w)v|_{\mathsf{v}}\} \\ &\le \log \prod_{\mathsf{v} \in M_K} \max\{1, \varepsilon_{\mathsf{v}}(Cl^{d-1})c_{\mathsf{v}}\} \\ &\le \log\left((Cl^{d-1})^{[K:\mathbb{Q}]}\right) + \log\left(\prod_{\mathsf{v} \in M_K} \max\{1, c_{\mathsf{v}}\}\right) \in O(\log l). \end{aligned}$$

Since $l = |w|$ this proves the claim.

(d) $\Rightarrow$ (a) Clearly $h(a_n) \in O(\log\log n)$ implies $h(a_n) \in o(\log n)$.

(b) $\Leftrightarrow$ (c) By (i) $\Leftrightarrow$ (ii) of Bell, Coons, and Hare [BCH16, Theorem 13] (which does not require the sequences to be $\mathbb{Z}$-valued). □

9. Fourth gap: characterization of automatic Mahler functions

In this section, we characterize Mahler functions $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ with $h(a_n) \in o(\log\log n)$, extending [BCH14, Theorem 1.1].

Theorem 9.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be a k-Mahler function. The following statements are equivalent.*

(a) *We have $h(a_n) \in o(\log\log n)$.*
(b) *For every minimal linear representation of $(a_n)_{n\ge0}$, the matrix semigroup $\mu(\Sigma_k^*)$ is finite.*
(c) *The power series $f(z)$ is k-automatic.*
(d) *We have $h(a_n) \in O(1)$. Equivalently, the set $\{ a_n : n \ge 0 \}$ is finite.*

The following lemma closely follows [BCH14, Lemma 2.1]

Lemma 9.2. *Let $(a_n)_{n\ge0}$ be a k-regular sequence in $\overline{\mathbb{Q}}$, with a minimal linear representation (u, μ, v). If $h(a_n) \in o(\log\log n)$, then the semigroup $\mu(\Sigma_k^*)$ is finite.*

Proof. Again $a_{[w]_k} = u\mu(w)v$ for all $w \in \Sigma_k^*$. By our assumption

$$h(u\mu(w)v) \in o(\log|w|)\,.$$

Now suppose to the contrary that $\mu(\Sigma_k^*)$ is infinite. A theorem of McNaughton and Zalcstein [MZ75] gives a positive answer to the strong Burnside problem for semigroups of matrices over a field. Since $\mu(\Sigma_k^*)$ is a finitely generated semigroup of matrices, but not finite, this theorem implies that there exists $w \in \Sigma_k^*$ such that $\mu(w^m) \ne \mu(w^n)$ for all m, $n \in \mathbb{Z}_{\ge0}$ with $m \ne n$. Fix such a word w.

Set $A \coloneqq \mu(w)$. By Lemma 8.4 every eigenvalue of A is either 0 or a root of unity. Our choice of w ensures that there exists at least one nonzero eigenvalue ζ with a non-trivial

Jordan block. Let $B \in \overline{\mathbb{Q}}^{d\times d}$ be an invertible matrix such that $B^{-1}AB$ is in Jordan normal form. Without restriction we may assume

$$B^{-1}AB = \begin{pmatrix} \zeta & 1 & 0 & \dots & 0 \\ 0 & \zeta & * & \dots & 0 \\ 0 & 0 & * & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & * \end{pmatrix}.$$

The $(1,2)$ entry of $B^{-1}A^nB$ is $n\zeta^{n-1}$. Hence $h(e_1^TB^{-1}A^nBe_2) = h(n\zeta^{n-1}) \geq \log n$.

Using the minimality of the linear representation, we can write $e_1^TB^{-1} = \sum_{i=1}^d \lambda_i u\mu(w_i)$ and $Be_2 = \sum_{i=1}^d \tau_i\mu(w_i')v$ with suitable λ_i, $\tau_i \in \overline{\mathbb{Q}}$ and w_i, $w_i' \in \Sigma_k^*$. It follows that

$$h\Big(\sum_{i,j=1}^d \lambda_i\tau_j u\mu(w_iw^nw_j')v\Big) \geq \log n\,.$$

Hence there exist i, $j \in \{1,\dots,d\}$ and $c \in \mathbb{R}_{>0}$ such that

$$h(u\mu(w_iw^nw_j')v) > c\log n \qquad \text{for infinitely many } n.$$

This is a contradiction to $h(u\mu(w_iw^nw_j')v) \in o(\log n)$. □

Proof of Theorem 9.1. Let $f(z) = \sum_{n=0}^\infty a_nz^n \in \overline{\mathbb{Q}}[\![z]\!]$ be a k-Mahler function.

(a) ⇒ (b) Let $h(a_n) \in o(\log\log n)$. Then $(a_n)_{n\geq 0}$ is k-regular by Theorem 7.1, with a minimal linear representation (u,μ,v). By Lemma 9.2 the semigroup $\mu(\Sigma_k^*)$ is finite.

(b) ⇒ (c) Let (u,μ,v) be a minimal linear representation of the regular sequence $(a_n)_{n\geq 0}$. Suppose $\mu(\Sigma_k^*)$ is finite. Then $(a_n)_{n\geq 0}$ takes only finitely many values, and k-regular sequences taking finitely many values are automatic ([AS03a, Theorem 16.1.5] or [BR11, Proposition 5.3.3]).

(c) ⇒ (d) Let $(a_n)_{n\geq 0}$ be k-automatic. Then the sequence only takes finitely many values by definition.

(d) ⇒ (a) Clearly, if $\{\, a_n : n \geq 0 \,\}$ is finite, then $h(a_n) \in o(\log\log n)$. □

10. Comments on Becker's conjecture

Every k-regular power series is k-Mahler, and as a partial converse Becker showed that a k-Becker power series is k-regular. He also conjectured a full description of k-regular power series in terms of k-Becker power series. This conjecture was recently proven by Bell, Chyzak, Coons, and Dumas, as the main result in [BCCD19]. The proof in [BCCD19] is stated for $K = \mathbb{C}$, but the same arguments apply equally well to arbitrary fields of characteristic zero.

Theorem 10.1 ([BCCD19, Theorem 1])**.** *Let K be a field of characteristic 0. If $f(z) \in K[\![z]\!]$ is k-regular, there exist a polynomial $q(z) \in K[z]$ with $q(0) = 1$ such that $1/q(z)$ is k-regular and a nonnegative integer γ such $f(z)/z^\gamma q(z)$ is a k-Becker Laurent series.*

By a *k-Becker Laurent series*, we of course mean a Laurent series satisfying a functional equation as in Definition 3.6. We stress that it is not always possible to obtain a k-Becker power series of the form $f(z)r(z)$ with $r(z)$ a rational function [BCCD19, Theorem 14].

The proof of Bell, Chyzak, Coons, and Dumas breaks down into two steps:

(I) First they show that a k-regular power series $f(z) \in K[\![z]\!]$ satisfies a k-Mahler equation
$$p_0(z)f(z) + p_1(z)f(z^k) + \cdots + p_d(z)f(z^{k^d}) = 0\,,$$
where all roots of $p_0(z)$ belong to $\{0\} \cup \mathcal{U}_k$. Equivalently, the k-denominator $\mathfrak{d}(z)$ of $f(z)$ has all its roots in $\{0\} \cup \mathcal{U}_k$.

(II) They show that any such series has the required decomposition.

We now give alternative arguments for both of these steps using our results. In particular, for $K = \overline{\mathbb{Q}}$, step I is immediate from Theorem 7.1 and our argument for step II is somewhat shorter. We also recover Proposition 2 and Corollary 3 of [BCCD19] (Corollary 3 follows as in the proof of Proposition 7.9).

10.1. **Step I.** For $K = \overline{\mathbb{Q}}$, Theorem 7.1 immediately establishes step I. We now show how to extend the relevant part of Theorem 7.1 to arbitrary fields of characteristic 0.

Let K be a field of characteristic 0 and let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in K[\![z]\!]$ be k-regular. Let $\mathfrak{d}(z)$ denote the Mahler denominator of $f(z)$ over K. We also have that the k-kernel of $f(z)$ generates a finite-dimensional K-vector space. In particular, there is some fixed $M > 0$ such that for every $j \in \{0, 1, \ldots, k^M - 1\}$, we have
$$a_{k^M n+j} = \sum_{e<M} \sum_{i=0}^{k^e-1} c_{j,i,e} a_{k^e n+i} \qquad \text{for } n \geq 0.$$
We have that for some $s \geq 0$, the power series $f(z), f(z^k), \ldots, f(z^{k^s})$ satisfy a Mahler system of the form Eq. (3) with some invertible matrix $A(z)$ with entries in $K(z)$. Let R be a finitely generated $\mathbb{Z}$-algebra that contains:

(1) $f(0), \ldots, f(k^M - 1)$;
(2) the roots of $\mathfrak{d}(z)$ and the reciprocals of all nonzero roots of $\mathfrak{d}(z)$;
(3) the structure constants $c_{i,j,e}$.
(4) the nonzero coefficients and their inverses of each polynomial appearing in either the numerator or denominator of an entry of $A(z)$.

Then by construction, $f(z) \in R[\![z]\!]$. If $\mathfrak{p}$ is a prime ideal of R and $g(z) \in R[\![z]\!]$, then we let $g_{|\mathfrak{p}}(z)$ denote the power series in $R/\mathfrak{p}[\![z]\!]$ obtained by reducing the coefficients of $g(z)$ modulo $\mathfrak{p}$. Then by construction, $f_{|\mathfrak{p}}(z)$ is a regular power series in $(R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p})[\![z]\!]$.

Lemma 10.2. *Let $\lambda \in R$ be a nonzero root of $\mathfrak{d}(z)$. Then λ is a root of unity.*

Proof. Suppose that λ is not root of unity. Then, since the coefficients of $A(z)$ have only finitely many poles, there is some n such that λ^{k^n} is a regular point with respect to this Mahler system. We now take a k-Mahler equation
$$q_0(z)f(z) = \sum_{i=1}^{L} q_i(z)f(z^{k^i}), \qquad q_0(z), q_1(z),\ \ldots\ ,\ q_L(z) \in K[z], \tag{12}$$
with $q_0(z) \neq 0$ and L minimal. Iterating Eq. (12) we find an equation
$$r_0(z)f(z) = \sum_{i=n}^{L+n-1} r_i(z)f(z^{k^i}), \tag{13}$$

where we may assume that the polynomials $r_0(z), r_n(z), \ldots, r_{L+n+1}(z) \in K[z]$ are coprime. The Mahler denominator $\mathfrak{d}(z)$ divides $r_0(z)$ and so $r_0(\lambda) = 0$. By coprimality of the coefficients, in particular there is some i_0 such that $r_{i_0}(\lambda) \neq 0$. Now we adjoin the coefficients of $r_0(z)$ and $r_n(z), \ldots, r_{n+L-1}(z)$ to R.

By Noether normalization, there is a positive integer N and $x_1, \ldots, x_d \in R$ such that $x_1, \ldots, x_d$ are algebraically independent over $\mathbb{Q}$ and such that $R[1/N]$ is a finite integral extension of $\mathbb{Z}[1/N][x_1, \ldots, x_d]$. Let $\mathcal{S}$ denote the set of prime ideals $\mathfrak{p}$ of $R[1/N]$ with the property that $\mathfrak{p} \cap \mathbb{Z}[1/N][x_1, \ldots, x_d] = (x_1 - b_1, \ldots, x_d - b_d)$ with $b_1, \ldots, b_d$ integers. By integrality, there is at least one such prime for each d-tuple $(b_1, \ldots, b_d)$ of integers. Moreover, $R/\mathfrak{p}$ is a finite extension of $\mathbb{Z}[1/N]$, generated by at most κ elements for some κ that is independent of $\mathfrak{p}$ (indeed, we may take κ to be the cardinality of the set of generators of $R[1/N]$ as a $\mathbb{Z}[1/N][x_1, \ldots, x_d]$-module). Hence $R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ is a number field of degree at most κ for each $\mathfrak{p} \in \mathcal{S}$.

Moreover, the intersection of the prime ideals in $\mathcal{S}$ is (0). For $\mathfrak{p} \in \mathcal{S}$, we let $\lambda_\mathfrak{p} = \lambda + \mathfrak{p} \in R/\mathfrak{p}$. Then we reduce Eq. (13) modulo $\mathfrak{p}$ and plug in $z = \lambda_\mathfrak{p}$ to obtain

$$0 = \sum_{i=n}^{L+n-1} r_{i|\mathfrak{p}}(\lambda_\mathfrak{p}) f_{|\mathfrak{p}}(\lambda_\mathfrak{p}^{k^i}), \tag{14}$$

where the left side follows from the fact that $\mathfrak{d}(z)$ divides $r_0(z)$. It is straightforward to see that $\lambda_\mathfrak{p}^{k^n} \in (R/\mathfrak{p})_\mathfrak{p}$ is a regular point of the reduced Mahler system for $f_{|\mathfrak{p}}(z), f_{|\mathfrak{p}}(z^k), \ldots, f_{|\mathfrak{p}}(z^{k^{L-1}})$ for $\mathfrak{p}$ in a Zariski dense subset $\mathcal{T}$ of $\mathcal{S}$. Moreover, there is a Zariski dense subset $\mathcal{T}'$ of $\mathcal{T}$ such that $r_{i_0|\mathfrak{p}}(\lambda_\mathfrak{p}) \neq 0$ for $\mathfrak{p} \in \mathcal{T}'$. We remark that there is a Zariski dense subset $\mathcal{T}''$ such that $\lambda_\mathfrak{p}$ is not a root of unity. To see this, observe that if $\mathfrak{p} \in \mathcal{T}'$ is such that $\lambda_\mathfrak{p}$ is a root of unity, then $\mathbb{Q}(\lambda_\mathfrak{p})$ is an extension of degree at most κ and hence there is some fixed $M = M(\kappa)$ such that $\lambda_\mathfrak{p}^M = 1$. Since $\mathcal{T}'$ is Zariski dense, this gives that λ is a root of unity, which is a contradiction.

Now for $\mathfrak{p} \in \mathcal{T}''$, we have $\lambda_\mathfrak{p} \in K_\mathfrak{p} := (R/\mathfrak{p})_\mathfrak{p}$. Then there is some place v on the number field $K_\mathfrak{p}$ such that $|\lambda_\mathfrak{p}|_\mathsf{v} < 1$. Equation (14) combined with Theorem 4.3 yields

$$0 = \sum_{i=n}^{L+n-1} r_{i|\mathfrak{p}}(z) f_{|\mathfrak{p}}(z^{k^i}).$$

By Zariski density of $\mathcal{T}''$, also $0 = \sum_{i=n}^{L+n-1} r_i(z) f(z^{k^i}) \in R[\![z]\!] \cap K[\![z]\!]$, in contradiction to the minimality of L. The result follows. $\square$

By looking at the asymptotic behavior on the unit circle we may once again strengthen the previous lemma.

Lemma 10.3. *Let $\lambda \in R$ be a nonzero root of $\mathfrak{d}(z)$. Then $\lambda \in \mathcal{U}_k$.*

Proof. Let $0 \neq \lambda$ be a root of $\mathfrak{d}(z)$. By the previous lemma $\lambda \in \mathcal{U}$. Therefore it suffices to show $\lambda^{k^j} \neq \lambda$ for all $j \geq 1$.

Suppose to the contrary that $\lambda^{k^j} = \lambda$ for some $j \geq 1$. Since R is finitely generated, it embeds into $\mathbb{C}$. Let $|\cdot|$ denote the induced absolute value on R. We may now conclude as in the proof of Proposition 7.8: from Proposition 7.7 we obtain $\log|a_n| \geq c \log^2 n$ infinitely often. However, using the linear representation of a k-regular sequence, we easily obtain $\log|a_n| \in O(\log n)$ as in Lemma 7.10 (or [AS92, Theorem 2.10]), a contradiction. ☐

We thus have the following theorem, extending a part of Theorem 7.1 to fields of characteristic 0 and also generalizing [BCCD19, Proposition 2].

Theorem 10.4. *Let K be a field of characteristic 0, let $f(z) \in K[\![z]\!]$ be k-Mahler, and let $\mathfrak{d}(z)$ be the Mahler denominator of $f(z)$. Then $f(z)$ is k-regular if and only if every non-zero root of $\mathfrak{d}(z)$ (in the algebraic closure $\overline{K}$) is a root of unity with order not coprime to k.*

Proof. If $f(z)$ is k-regular, the claim follows from the previous lemma. The converse direction follows exactly as in the proof of Proposition 7.9. ☐

10.2. **Step II.** We now provide a somewhat shorter argument for the second step of [BCCD19]. First recall the following easy lemma.

Lemma 10.5. *Let K be a field and let $f(z) \in K[\![z]\!]$ be a k-Mahler power series solution to the equation*

$$p_0(z)f(z) + p_1(z)f(z^k) + \cdots + p_d(z)f(z^{k^d}) = 0\,. \tag{15}$$

If there exists a polynomial $q(z)$ such that $p_0(z)q(z)$ divides $q(z^{k^j})$ for all j, $1 \le j \le d$, then $f(z)/q(z) \in K(\!(z)\!)$ is a k-Becker Laurent series.

Proof. Set $g(z) := f(z)/q(z)$, then (15) gives

$$p_0(z)q(z)g(z) + p_1(z)q(z^k)g(z^k) + \cdots + p_d(z)q(z^{k^d})g(z^{k^d}) = 0\,.$$

Thus, $g(z) = -\sum_{i=1}^d r_i(z)g(z^{k^i})$, where $r_i(z) = p_i(z)q(z^{k^i})/(p_0(z)q(z)) \in K[z]$. ☐

Proof of Becker's conjecture (Theorem 10.1). Since $f(z)$ is k-regular, we know, by the first step, that $f(z)$ satisfies an equation of the form

$$p_0(z)f(z) + p_1(z)f(z^k) + \cdots + p_d(z)f(z^{k^d}) = 0\,,$$

where all roots of $p_0(z)$ belong to $\{0\} \cup \mathcal{U}_k$. Thus, every nonzero root is a primitive ℓ-root of unity for some ℓ not coprime with k. For such a natural number ℓ, there exist a positive integer r and a nonnegative integer s such that $\gcd(\ell, k^j) = r$ for all $j > s$. Let s be minimal with this property. Let A denote the set of nonzero roots of $p_0(z)$, and, for $\xi \in A$, set $a(\xi) := \ell(\xi)/r(\xi)$. Let $\phi_n(z)$ denote the nth cyclotomic polynomial. Then $\phi_{\ell(\xi)}(z)\phi_{a(\xi)}(z^{k^s})$ divides $\phi_{a(\xi)}(z^{k^{s+j}})$ for all $j \ge 1$. In particular, $(z-\xi)\phi_{a(\xi)}(z^{k^s})$ divides $\phi_{a(\xi)}(z^{k^{s+j}})$ for all j, $1 \le j \le m$. Setting

$$q(z) := \prod_{\xi \in A} \phi_{a(\xi)}(z^{k^s})\,,$$

and applying Lemma 10.5, we obtain that $f(z)/z^\gamma q(z)$ is a k-Becker Laurent series, where γ is the valuation of $p_0(z)$. Moreover, $1/q(z)$ is a k-Becker power series for

$$q(z)^{-1} = \frac{q(z^k)}{q(z)} \cdot q(z^k)^{-1}$$

and by construction $q(z)$ divides $q(z^k)$. In particular, $1/q(z)$ is k-regular. ☐

11. Automatic Mahler power series over arbitrary fields

We first show how to extend our characterization of k-automatic Mahler functions to arbitrary ground fields of characteristic zero.

Theorem 11.1. *Let K be a field of characteristic 0 and let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in K[\![z]\!]$ be a k-Mahler power series. Then $(a_n)_{n\geq 0}$ is k-automatic if and only if $\{ a_n : n \geq 0 \}$ is finite.*

In order to prove Theorem 11.1, we use a standard specialization argument.

Lemma 11.2. *Let K be a field of characteristic zero containing $\overline{\mathbb{Q}}$, and let $u_1, \ldots, u_d \in K$. Then there exists a ring homomorphism $\varphi\colon \overline{\mathbb{Q}}[u_1, \ldots, u_d] \to \overline{\mathbb{Q}}$ leaving $\overline{\mathbb{Q}}$ invariant.*

Proof. This is an easy consequence of the weak Nullstellensatz. A proof can be found in [EG15, Lemma 6.3.3]. □

Proof of Theorem 11.1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in K[\![z]\!]$ be a k-Mahler power series with finite set of coefficients. Replacing K by its algebraic closure, we may assume $\overline{\mathbb{Q}} \subseteq K$. Let $p_0(z), \ldots, p_d(z) \in K[z]$ be such that

$$p_0(z)f(z) + p_1(z)f(z^k) + \cdots + p_d(z)f(z^{k^d}) = 0\,.$$

Let C be the finite set consisting of all coefficients of $f(z)$ and $p_0(z), \ldots, p_d(z)$. We apply Lemma 11.2 with the set $\{u_1, \ldots, u_d\}$ consisting of all $c \in C$, all $c - d$ with $c, d \in C$, as well as the inverses of all these elements that are nonzero. Thus $\varphi(c) \neq 0$ for $c \neq 0$ and $\varphi(c) \neq \varphi(d)$ for $c \neq d$. The resulting homomorphism extends to $\varphi\colon \overline{\mathbb{Q}}[u_1, \ldots, u_m][\![z]\!] \to \overline{\mathbb{Q}}[\![z]\!]$, and

$$\varphi(p_0)(z) \cdot \varphi(f)(z) + \varphi(p_1)(z) \cdot \varphi(f)(z^k) + \cdots + \varphi(p_d)(z) \cdot \varphi(f)(z^{k^d}) = 0$$

is a k-Mahler equation for $\varphi(f)(z)$. Thus Theorem 9.1 implies that the sequence $(\varphi(a_n))_{n\geq 0}$ is k-automatic. Since $\varphi\colon C \to \overline{\mathbb{Q}}$ is injective, the same is true for $(a_n)_{n\geq 0}$. □

11.1. **The case of a base field of positive characteristic.** Theorem 11.1 strongly depends on the characteristic of the field being zero. If K is a field of characteristic $p > 0$, we still have a similar result for p-Mahler power series (see Proposition 11.3), but if k is coprime to p this is no longer true. Indeed, the series

$$\prod_{i=0}^{\infty} (1 - z^{k^i})^{-1} \in K[\![z]\!]$$

is not k-automatic by [Bec94, Proposition 1], despite being k-Mahler with coefficients taking only finitely many values (because they belong to the prime field).

Proposition 11.3. *Let K be a field of characteristic p and let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in K[\![z]\!]$ be a k-Mahler power series where k is a power of p. Then the sequence $(a_n)_{n\geq 0}$ is k-automatic if and only if $\{ a_n : n \geq 0 \}$ is finite.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in K[\![z]\!]$ be p^m-Mahler for some positive integer m. Let us assume that $\{ a_n : n \geq 0 \}$ is finite. Let us consider a non-trivial equation

$$p_0(z)f(z) + p_1(z)f(z^{p^m}) + \cdots + p_d(z)f(z^{p^{md}}) = 0\,.$$

We let R denote the finitely generated $\mathbb{F}_p$-algebra generated by the coefficients a_n, the inverses of all nonzero differences $a_i - a_j$, and the coefficients of the polynomials $p_i(z)$, as well as the inverses of their nonzero coefficients.

Let $\mathfrak{M}$ be some maximal ideal of R. Then $R/\mathfrak{M} = \mathbb{F}_q$ with q a power of p, say $q = p^\ell$. Let $f_{|\mathfrak{M}}(z) := \sum_{n=0}^{\infty}(a_n \bmod \mathfrak{M})z^n$ denote the reduction of $f(z)$ modulo $\mathfrak{M}$. Then $f_{|\mathfrak{M}}(z)$ is p^m-Mahler and hence it is also $p^{m\ell}$-Mahler by Lemma 3.2. Thus, we deduce that $f_{|\mathfrak{M}}(z)$ is algebraic over $\mathbb{F}_q(z)$. By Christol's theorem (see [AS03a, Chapter 12]), the sequence $(a_n \bmod \mathfrak{M})_{n\geq 0}$ is p-automatic. But by definition of R, if $a_i \neq a_j$ then $a_i \bmod \mathfrak{M} \neq a_j \bmod \mathfrak{M}$. Thus the sequence $(a_n)_{n\geq 0}$ is also p-automatic, and hence p^m-automatic. □

12. Decidability

A k-Mahler function can be uniquely specified by the finite data consisting of a k-Mahler equation it satisfies and sufficiently many initial coefficients of the power series. Therefore it is reasonable to ask whether, for a given k-Mahler function, it can be decided which of the five cases of Theorem 1.1 it falls into. However, we neither try to describe an efficient algorithm to perform this task, nor do we provide an upper bound for the complexity of the algorithm that could be extracted from what follows.

Theorem 12.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be a k-Mahler function (specified by a k-Mahler equation and sufficiently many initial coefficients). Then it is decidable which of the five growth classes in Theorem 1.1 the function f falls into.*

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be k-Mahler. As Theorems 6.1 and 7.1 show, the minimal denominator $\mathfrak{d}(z) \in \overline{\mathbb{Q}}[z]$ of $f(z)$ plays a crucial role in determining the growth class that $f(z)$ falls into: the growth depends on whether $\mathfrak{d}(z)$ has roots outside $\{0\} \cup \mathcal{U}$, respectively outside $\{0\} \cup \mathcal{U}_k$. This raises the question whether there is an effective way of deciding which of the three cases occurs. Along similar lines, if $f(z)$ is k-regular, the question arises whether it is decidable into which of the three cases (k-regular, $\overline{\mathbb{Q}}$-linear combination of word-convolution products of k-automatic sequences, and k-automatic) the coefficients of $f(z)$ falls. In this section, we establish that all these properties are decidable.

Suppose $f(z)$ is specified by a k-Mahler equation and sufficiently many initial coefficients to determine the solution uniquely. Then we can compute any finite number of initial coefficients by recursion. By work of Adamczewski and Faverjon [AF18] it is possible to find a *minimal* (homogeneous) k-Mahler equation, that is, polynomials $p_0(z), p_1(z), \ldots, p_d(z) \in K[z]$, where K is a number field,

$$p_0(z)f(z) + p_1(z)f(z^k) + \cdots + p_d(z)f(z^{k^d}) = 0\,, \tag{16}$$

where d is minimal and $p_0(z), \ldots, p_d(z)$ are coprime.

By definition $\mathfrak{d}(z)$ divides $p_0(z)$. It is tempting to hope that, to determine the types of roots of $\mathfrak{d}(z)$, it suffices to consider those of $p_0(z)$. Unfortunately, this hope is thwarted by Example 3.10. We can however still determine the types of roots of $\mathfrak{d}(z)$.

Proposition 12.2. *There exists an algorithm to determine whether the k-Mahler denominator $\mathfrak{d}(z)$ of a k-Mahler function $f(z)$ has a root outside $\{0\} \cup \mathcal{U}$.*

Moreover, if all roots of $\mathfrak{d}(z)$ *are contained in* $\{0\} \cup \mathcal{U}$*, then we can find an explicit* k*-Mahler equation*

$$q_0(z)f(z) = q_1(z)f(z^{k^{n_0}}) + \cdots + q_d(z)f(z^{k^{n_0+d-1}})$$

with $n_0 \geq 1$*, with* $q_0(z), \ldots, q_d(z) \in \overline{\mathbb{Q}}[z]$ *and all roots of* $q_0(z)$ *contained in* $\{0\} \cup \mathcal{U}$*.*

Proof. Let us consider the minimal equation (16). By [AF18], this equation can be explicitly determined (this is a variation of Algorithm 1.3 in [AF18]). We may assume that the number field K contains all coefficients and roots of $p_0(z), \ldots, p_d(z)$. Set

$$\mathcal{S} := \{\, \lambda : p_0(\lambda)p_d(\lambda) = 0 \,\}$$

and

$$\rho = \min_{\mathsf{v} \in \mathcal{M}_K} \{ \min\{ |\lambda|_{\mathsf{v}} : p_0(\lambda)p_d(\lambda) = 0 \} \} .$$

Now, let n_0 be the minimal positive integer such that $|\lambda^{k^{n_0}}|_{\mathsf{v}} < \rho$ for all λ in $\mathcal{S}$ and all places v such that $|\lambda|_{\mathsf{v}} < 1$ (there are only a finite number of such places). The integer n_0 can be explicitly determined. By repeated substitution, we can explicitly determine an equation

$$q_0(z)f(z) = q_1(z)f(z^{k^{n_0}}) + \cdots + q_d(z)f(z^{k^{n_0+d-1}}) , \tag{17}$$

for $f(z)$. Suppose first that $q_0(z)$ does *not* have a root $0 \neq \lambda$ that is not a root of unity. Then neither does $\mathfrak{d}(z)$, because $\mathfrak{d}(z)$ divides $q_0(z)$.

Suppose now $q_0(z)$ has a root $0 \neq \lambda$ that is not a root of unity. By Kronecker's theorem, there exists a place v such that $0 < |\lambda|_{\mathsf{v}} < 1$. Arguing exactly as in the proof of Proposition 6.4, we see that λ is a pole of $f(z)$. Thus $f(z)$ has a radius of convergence strictly less than 1 with respect to $|\cdot|_{\mathsf{v}}$. By Theorem 6.1 also $\mathfrak{d}(z)$ must have a nonzero root that is not a root of unity. □

Assuming $\mathfrak{d}(z)$ does not have a root outside of $\{0\} \cup \mathcal{U}$, we now want to determine if it has a root in $\mathcal{U} \setminus \mathcal{U}_k$.

Lemma 12.3. *Let* $d \in \mathbb{Z}_{>0}$*, let* $0 \neq \zeta \in \mathbb{C}$ *such that* $\zeta^k = \zeta$*, and let* $A(z) \in \overline{\mathbb{Q}}(z)^{d\times d}$*. Assume that* $w(z) \in \mathbb{C}[\![z]\!]^d$ *satisfies the equation*

$$w(z) = A(z)w(z^k) .$$

Assume also that the following properties hold.

(i) *The coordinates of* $A(z)$ *have no poles at* ζ *and no poles in* $B(0,1)$*.*
(ii) *The coordinates of* $w(z)$ *are continuous in* $B(0,1)$*.*

Then, there exists $c \in \mathbb{R}_{>0}$ *such that*

$$\|w(t\zeta)\| < |1-t|^{-c} \qquad \text{for all } t \in (0,1).$$

Proof. Since the map $[0,1] \to \mathbb{C}^{d\times d}$, $t \mapsto A(t\zeta)$ is continuous, there exists $c_0 \geq 1$ such that $\|A(t\zeta)\| \leq c_0$ for all $t \in [0,1]$. Let $\varepsilon \in (0,1)$ and $c_1 = \max\{\, \|w(t\zeta)\| : t \in [0,\varepsilon] \,\}$.

Let $t \in [0,1)$, and let $n \in \mathbb{Z}_{\geq 0}$ be minimal such that $t^{k^n} \leq \varepsilon$. We can obtain an upper bound on n as follows. The inequality $t^{k^n} \leq \varepsilon$ is equivalent to $k^n \log t \leq \log \varepsilon$, which is equivalent to $k^n(-\log t) \geq -\log \varepsilon$. In turn, this is equivalent to $n + \log_k(-\log t) \geq \log_k(-\log \varepsilon)$. So

$$n = \lceil \log_k(-\log \varepsilon) - \log_k(-\log t) \rceil .$$

Thus
$$n \le c_2 - \log_k(-\log t) \qquad \text{with } c_2 = 1 + \log_k(-\log \varepsilon)\,.$$

Now
$$k^n \le k^{c_2} k^{-\log_k(-\log t)} \le k^{c_2} \frac{1}{-\log t} \le k^{c_2} \frac{1}{1-t}\,,$$
where we used $\log t \le t-1$ for the last inequality. We have
$$w(t\zeta) = A(t\zeta)A(t^k\zeta)\cdots A(t^{k^{n-1}}\zeta)w(t^{k^n}\zeta)\,,$$
and thus $\|w(t\zeta)\| \le c_0^n c_1$. Now
$$c_0^n c_1 = c_1 k^{n\log_k c_0} \le c_1 k^{c_2 \log_k c_0}(1-t)^{-\log_k c_0}.$$
The constant may be absorbed by replacing the exponent by a bigger one. ☐

Proposition 12.4. *Let $f(z) \in \overline{\mathbb{Q}}[\![z]\!]$ be k-Mahler with k-Mahler denominator $\mathfrak{d}(z)$. Suppose all roots of $\mathfrak{d}(z)$ are contained in $\{0\} \cup \mathcal{U}$. There exists an algorithm to decide whether $\mathfrak{d}(z)$ has a root in $\mathcal{U} \setminus \mathcal{U}_k$.*

Moreover, if all roots of $\mathfrak{d}(z)$ are contained in $\{0\} \cup \mathcal{U}_k$, then we can find an explicit k-Mahler equation
$$s_0(z)f(z) = s_1(z)f(z^k) + \cdots + s_d(z)f(z^{k^d})$$
with $s_0(z), \ldots, s_d(z) \in \overline{\mathbb{Q}}[z]$ and all roots of $s_0(z)$ contained in $\{0\} \cup \mathcal{U}_k$.

Proof. Let $\mathfrak{d}(z)f(z) = p_1(z)f(z^k) + \cdots + p_d(z)f(z^{k^d})$ with $p_1(z), \ldots, p_d(z) \in \overline{\mathbb{Q}}[z]$. Using the condition on $\mathfrak{d}(z)$ together with the fact that $f(z)$ converges in a neighborhood of 0, this equation implies that $f(z)$ is analytic in $B_{|\cdot|}(0,1)$ for every absolute value $|\cdot|$ on $\overline{\mathbb{Q}}$.

Now let $q_0(z)f(z) = q_1(z)f(z^k) + \cdots + q_d(z)f(z^{k^d})$ with $q_0(z), \ldots, q_d(z) \in \overline{\mathbb{Q}}[z]$ and $q_0(z)q_d(z) \ne 0$ be an explicit k-Mahler equation for $f(z)$. Since $\mathfrak{d}(z)$ divides $q_0(z)$, we only have to check if any of the finitely many roots of $q_0(z)$ in $\mathcal{U} \setminus \mathcal{U}_k$ are roots of $\mathfrak{d}(z)$.

Suppose ζ is such a root of $q_0(z)$. Then there exists an, explicitly determinable, integer $j_0 \ge 1$ such that $\zeta^{k^{j_0}} = \zeta$. Let $\ell = k^{j_0}$. Again using [AF18] we can find an ℓ-Mahler equation for $f(z)$, say

$$r_0(z)f(z) = r_1(z)f(z^\ell) + \cdots + r_e(z)f(z^{\ell^e}) \tag{18}$$

with $r_0(z), \ldots, r_e(z) \in \overline{\mathbb{Q}}[z]$ coprime and $r_0(z)r_e(z) \ne 0$. If $r_0(\zeta) \ne 0$, then $\mathfrak{d}(\zeta) \ne 0$.

Suppose now $r_0(\zeta) = 0$. We will show $\mathfrak{d}(\zeta) = 0$. Fix any embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and thereby an archimedean absolute value $|\cdot|$ on $\overline{\mathbb{Q}}$. Proposition 7.7 implies that there exist $a, b, c \in \mathbb{R}_{>0}$, $m_0, n_0 \in \mathbb{Z}_{\ge 0}$, and a sequence $(t_j)_{j\ge 0} \to 1$ in $[0,1)$ such that
$$\Big|\sum_{n=0}^{\infty} a_{n+n_0}(\zeta t_j)^n\Big| \ge (1-t_j)^a \exp(b \log^2 m) t_j^{mc} \qquad \text{for all } j \ge 0 \text{ and } m \ge m_0.$$
Define $f_0(z) = \sum_{n=0}^{\infty} a_{n+n_0} z^n$ and $m_j = \lceil 1/(1-t_j) \rceil$. Then
$$\log|f_0(\zeta t_j)| \ge a\log(1-t_j) + b\log^2\lceil 1/(1-t_j)\rceil + \lceil 1/(1-t_j)\rceil c \log t_j\,.$$
As in the proof of Lemma 7.3 we see that the right side is asymptotically equivalent to $b\log^2(1-t_j)$. Now, if we had $\mathfrak{d}(\zeta) \ne 0$, then Lemma 12.3 would give $\log|f_0(\zeta t_j)| \le -c\log(1-t_j)$ for some $c \in \mathbb{R}_{>0}$, a contradiction.

To explicitly find an equation with $s_0(z)$ as desired, note that for each $\zeta \in \mathcal{U} \setminus \mathcal{U}_k$ that is a root of $q_0(z)$, we have found some k-Mahler equation, Eq. (18), for $f(z)$ with $r_0(\zeta) \neq 0$. Taking the greatest common divisor of $q_0(z)$ and all these $r_0(z)$ as ζ varies over the roots, we obtain the desired equation. $\square$

We have now shown that it is possible to decide algorithmically which of Cases (1) and (2) of Theorem 1.1 a given Mahler function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ falls into. Suppose now that $f(z)$ is k-regular. In this case, we wish to also decide whether $f(z)$ belongs to class (3), (4), or (5) of Theorem 1.1.

12.1. **From k-Mahler equations to linear representations.** We have represented an arbitrary k-Mahler function $f(z)$ by a k-Mahler equation and sufficiently many initial coefficients. If $f(z)$ is k-regular, it is more natural to represent the sequence of coefficients by a linear representation. We show that such a linear representation is computable from a k-Mahler equation satisfied by $f(z)$. Recall that Δ_r with $r \in \Sigma_k$ denotes the Cartier operators (Definition 6.3).

Lemma 12.5. *Let $f_1(z), \dots, f_d(z) \in \overline{\mathbb{Q}}[\![z]\!]$ with $f_i(z) = \sum_{n=0}^{\infty} a_{i,n} z^n$. Suppose that, for every $r \in \Sigma_k$ and every $1 \leq i \leq d$, there are explicitly known coefficients $\lambda_{r,1}, \dots, \lambda_{r,d} \in \overline{\mathbb{Q}}$ such that*

$$\Delta_r(f_i(z)) = \lambda_{r,1,i} f_1(z) + \cdots + \lambda_{r,d,i} f_d(z)\,.$$

Then we get an explicit linear representation for the k-regular sequence $(a_{1,n})_{n\geq 0}$.

Proof. Let $\mu\colon \Sigma_k^* \to \overline{\mathbb{Q}}^{d\times d}$ be defined by

$$\mu(r) \coloneqq \begin{pmatrix} \lambda_{r,1,1} & \dots & \lambda_{r,1,d} \\ \vdots & \ddots & \vdots \\ \lambda_{r,d,1} & \dots & \lambda_{r,d,d} \end{pmatrix} \qquad \text{and let} \qquad \mathbf{a}(n) \coloneqq (a_{1,n}, \dots, a_{d,n}).$$

Since $\Delta_r(f_i(z)) = \sum_{n=0}^{\infty} a_{i,kn+r} z^n$, we obtain $\mathbf{a}(kn+r) = \mathbf{a}(n)\mu(r)$ for $r \in \Sigma_k$. Finally let $e_1 = (1, 0, \dots, 0)^T \in \overline{\mathbb{Q}}^{d\times 1}$. Then $a_{1,[w]_k} = \mathbf{a}([w]_k)e_1 = \mathbf{a}(0)\mu(w)e_1$ for all words $w \in \Sigma_k^*$. $\square$

Lemma 12.6. *Let $p_1(z), \dots, p_d(z) \in \overline{\mathbb{Q}}[z]$ with $e = \max\{\deg p_1(z), \dots, \deg p_d(z)\}$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ satisfies the k-Becker equation*

$$f(z) = p_1(z) f(z^k) + \cdots + p_d(z) f(z^{k^d})\,,$$

then a linear representation for the k-regular sequence $(a_n)_{n\geq 0}$ is computable from a_0 and $p_1(z), \dots, p_d(z)$.

Proof. Following Becker [Bec94, Theorem 2], we see that the $\overline{\mathbb{Q}}$-vector space V spanned by $\{\, z^i f(z^{k^j}) : 0 \leq i \leq e,\ 0 \leq j \leq d \,\}$ is closed under all Cartier operators. Explicitly, if $r \in \Sigma_k$, $0 \leq i \leq e$, and $j \geq 1$, then

$$\Delta_r(z^i f(z^{k^j})) = \Delta_r(z^i) f(z^{k^{j-1}}) \in V,$$

since $\Delta_r(z^i) = z^{(i-r)/k}$ if $i \equiv r \mod k$ and $\Delta_r(z^i) = 0$ otherwise. If $j = 0$, then $\deg(\Delta_r(z^i p_j(z))) \leq 2e/k \leq e$, and thus

$$\Delta_r(z^i f(z)) = \Delta_r\Big(\sum_{j=1}^{d} z^i p_j(z) f(z^{k^j})\Big) = \sum_{j=1}^{d} \Delta_r(z^i p_j(z)) f(z^{k^{j-1}}) \in V\,.$$

Since $\Delta_r(z^i p_j(z))$ can be explicitly computed, we may apply Lemma 12.5 to find a linear representation of $(a_n)_{n\geq 0}$. Since $0^i f(0^{k^j}) \in \{0, a_0\}$, the resulting linear representation only depends on $p_1(z), \dots, p_d(z)$ and a_0. □

It is rather non-trivial that the convolution product of k-regular sequences is again k-regular. The standard way to show this uses the module-theoretic characterization of k-regularity; see [AS03a, Theorem 16.4.1] or [BR11, Proposition 5.2.7]. To see that a linear representation of the convolution product is computable from linear representations, we need to revisit this proof.

Remark 12.7. Using the growth-based characterization of k-regular sequences in Theorem 6.1, it is easy to show that the convolution product of k-regular sequences is k-regular. However, since this characterization already makes use of this fact that convolution products of k-regular sequences are k-regular (in Proposition 7.9), this does not actually give a new, independent proof.

Lemma 12.8. *Let $(a(n))_{n\geq 0}$ and $(b(n))_{n\geq 0}$ be two k-regular sequences in $\overline{\mathbb{Q}}$, each being given by a linear representation. Then a linear representation of the convolution product $(a \star b(n))_{n\geq 0}$ is computable.*

Let us recall that by definition $a \star b(n) = \sum_{i=0}^{n} a(i)b(n-i)$.

Proof. For $r \in \Sigma_k$, let $(\Delta_r(a)(n))_{n\geq 0}$ be the sequence defined by $\Delta_r(a)(n) \coloneqq a(kn + r)$. The key step in the proof of Allouche and Shallit [AS03a, Theorem 16.4.1] is the reduction (with $r \in \Sigma_k$)

$$\Delta_r(a \star b)(n) = \sum_{0\leq s\leq r} \Delta_s(a) \star \Delta_{r-s}(b)(n) + \sum_{r<s\leq k-1} \Delta_s(a) \star \Delta_{k+r-s}(b)(n-1)\,. \tag{19}$$

We will adapt this proof to the case where a and b are given by linear representations. Without restriction we can assume that the linear representations of a and b both have the same dimension $d \geq 0$. After a change of basis, we may take the linear representation of a to be $(\mathbf{a}(0), \kappa, e_1)$, where $\mathbf{a}(0) = (a_1(0), \dots, a_d(0))$ with $a_1(0) = a(0)$, where $e_1 = (1, 0, \dots, 0)^T$, and where $\kappa\colon \Sigma_k^* \to \overline{\mathbb{Q}}^{d\times d}$ is a monoid homomorphism. Define $\mathbf{a}(n) \coloneqq \mathbf{a}(0)\kappa(\langle n\rangle_k)$, where $\langle n\rangle_k \in \Sigma_k^*$ is the canonical base-k expansion of n. Then, in particular,

$$(a_1(kn + r), \dots, a_d(kn + r)) = (a_1(n), \dots, a_d(n))\, \kappa(r) \qquad \text{for } r \in \Sigma_k.$$

For b we have a linear representation $(\mathbf{b}(0), \lambda, e_1)$ with analogous definitions.

We construct a linear representation for $a \star b$ of dimension $2d^2$. For this, we index the first set of d^2 coordinates by (i, j) in lexicographic order, and the second by (i', j'), where $1 \leq i, j \leq d$. That is, the coordinates are indexed by $(1, 1)$, $(1, 2)$, …, (d, d), $(1', 1')$, $(1', 2')$, …, (d', d'). We use subscripts to indicate the corresponding entry of a matrix, e.g., $\kappa(r)_{i,j}$ is the entry in the i-th row and j-column of $\kappa(r)$. For $1 \le i,j \le d$ and r, $s \in \Sigma_k$, we get

$$\Delta_r(a_i) \star \Delta_s(b_j) = \Big(\sum_{\ell=1}^{d} a_\ell \kappa(r)_{\ell,i}\Big) \star \Big(\sum_{m=1}^{d} b_m \lambda(s)_{m,j}\Big) = \sum_{\ell,m=1}^{d} \kappa(r)_{\ell,i}\lambda(s)_{m,j}(a_\ell \star b_m)\,.$$

Using Eq. (19),

$$\begin{aligned}\Delta_r(a_i \star b_j)(n) &= \sum_{0\le s\le r}\sum_{\ell,m=1}^{d} \kappa(s)_{\ell,i}\lambda(r-s)_{m,j}\,(a_\ell \star b_m)(n)\\ &\quad+ \sum_{r<s\le k-1}\sum_{\ell,m=1}^{d} \kappa(s)_{\ell,i}\lambda(k+r-s)_{m,j}\,(a_\ell \star b_m)(n-1)\\ &= \sum_{\ell,m=1}^{d}\Big(\sum_{0\le s\le r}\kappa(s)_{\ell,i}\lambda(r-s)_{m,j}\Big)(a_\ell \star b_m)(n)\\ &\quad+ \sum_{\ell,m=1}^{d}\Big(\sum_{r<s\le k-1}\kappa(s)_{\ell,i}\lambda(k+r-s)_{m,j}\Big)(a_\ell \star b_m)(n-1)\,.\end{aligned}$$

Further note if $r \ge 1$, then $a_i \star b_j(kn+r-1) = \Delta_{r-1}(a_i \star b_j)(n)$. For $r=0$ we have $a_i \star b_j(kn-1) = a_i \star b_j(k(n-1)+(k-1)) = \Delta_{k-1}(a_i \star b_j)(n-1)$. In this case, in Eq. (19), the second sum vanishes, and we again obtain $a_i \star b_j(kn-1)$ as a linear combination of the $a_\ell \star b_m(n-1)$, namely,

$$\Delta_{k-1}(a_i \star b_j)(n-1) = \sum_{\ell,m}^{d}\Big(\sum_{0\le s\le k-1}\kappa(s)_{\ell,i}\lambda(k-1-s)_{m,j}\Big)(a_\ell \star b_m)(n-1).$$

For two $d \times d$-matrices A, B, the Kronecker product $A \otimes B$ is the $d^2 \times d^2$-matrix defined by $(A \otimes B)_{(i,j),(\ell,m)} = A_{i,\ell}B_{j,m}$. For $0 \ne r \in \Sigma_k$, we define the $2d^2 \times 2d^2$-matrix $\mu(r)$ by the block structure

$$\mu(r) := \begin{pmatrix} \sum_{0\le s\le r}\kappa(s)\otimes\lambda(r-s) & \sum_{0\le s\le r}\kappa(s)\otimes\lambda(r-1-s)\\ \sum_{r<s\le k-1}\kappa(s)\otimes\lambda(k+r-s) & \sum_{r<s\le k-1}\kappa(s)\otimes\lambda(k+r-1-s)\end{pmatrix}.$$

Similarly

$$\mu(0) := \begin{pmatrix} \kappa(0)\otimes\lambda(0) & 0\\ \sum_{s=1}^{k-1}\kappa(s)\otimes\lambda(k-s) & \sum_{s=0}^{k-1}\kappa(s)\otimes\lambda(k-1-s)\end{pmatrix}.$$

Define for each $n \ge 0$ the $2d^2$ row vector $\mathbf{v}(n)$ by $\mathbf{v}(n)_{(\ell,m)} = a_\ell \star b_m(n)$ and $\mathbf{v}(n)_{(\ell',m')} = a_\ell \star b_m(n-1)$. Then

$$\mathbf{v}(kn+r) = \mathbf{v}(n)\mu(r).$$

Now $\mathbf{v}_{(1,1)}(n) = a_1 \star b_1(n) = a \star b(n)$. Thus, the triple $(\mathbf{v}(0), \mu, e_{(1,1)})$, where $e_{(1,1)}$ is the $2d^2$ column vector with 1 in the coordinate $(1,1)$ and zeroes everywhere else, is a linear representation for $a \star b$. $\square$

Proposition 12.9. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be k-regular, given by a k-Mahler equation, the minimal $n_0 \ge 0$ with $a_{n_0} \ne 0$, and the value a_{n_0}. Then a linear representation for the k-regular sequence $(a_n)_{n\ge0}$ is computable.*

Proof. From a linear representation of $(a_n)_{n\geq n_0}$ it is easy to find one for $(a_n)_{n\geq 0}$. We may therefore without restriction assume $n_0 = 0$. (An explicit k-Mahler equation for this power series can be found using [AB17, Lemma 6.1].)

Using Propositions 12.2 and 12.4, we can further find a k-Mahler equation

$$p_0(z)f(z) = p_1(z)f(z^k) + \cdots + p_d f(z^{k^d}),$$

with $p_0(z), \dots p_d(z) \in \overline{\mathbb{Q}}[z]$, with $p_0(z)$ and $p_d(z)$ coprime, and with the property that all roots of $p_0(z)$ are contained in $\mathcal{U}_k$. In particular, we may assume $p_0(0) = 1$.

Now Theorem 3.8 gives a decomposition

$$f(z) = g(z)\Big(\prod_{i=0}^{\infty} p_0(z^{k^i})\Big)^{-1},$$

where $g(z)$ is k-Becker, and a k-Becker equation for $g(z)$ can be computed. Lemma 12.6 yields a linear representation for the coefficient sequence of $g(z)$. Factoring $p_0(z)$ into linear factors of the form $1 - z\zeta^{-1}$ with $\zeta \in \mathcal{U}_k$, we recall that $\prod_{i=0}^{\infty}(1 - z^{k^i}\zeta^{-1})^{-1}$ is k-regular. Indeed, by [AB17, Proposition 7.8], this infinite product factors as a polynomial and a k-Becker function (both computable). Using Lemma 12.8 we find a linear representation for $\prod_{i=0}^{\infty}(1 - z^{k^i}\zeta^{-1})^{-1}$. Finally, Lemma 12.8 allows us to find a linear representation for $f(z)$ itself. □

12.2. **Tame and finite semigroups.** From a linear representation, a minimal linear representation is computable, and we may now assume that the k-regular sequence $(a_n)_{n\geq 0}$ is given by such a minimal linear representation (u, μ, v). To decide which of Cases (3)–(5) of Theorem 1.1 the sequence belongs to, it now suffices to decide whether or not the finitely generated matrix semigroup $\mu(\Sigma_k^*)$ is finite, respectively, tame.

For this, we first need the following two lemmas.

Lemma 12.10. *Let $A_1, \dots, A_t \in \overline{\mathbb{Q}}^{d\times d}$. It is possible to decide whether or not the matrices $A_1, \dots, A_t$ have a proper nonzero common invariant subspace, and if so, to compute one.*

Proof. This can be done using exterior powers and Gröbner bases, see Arapura and Peterson [AP04]. A model-theoretic approach is given by Pastuszak in [Pas17]. Both papers discuss the history of this problem. □

Lemma 12.11. *Let K be a number field and $d \geq 0$. For every $r \geq 0$, there exists a computable $n = n(r, K)$ with the following property: if $\mathcal{S} \subseteq K^{d\times d}$ is a* finite *semigroup generated by r matrices, then $\#\mathcal{S} \leq n$.*

Proof. By a result of Mandel and Simon [MS78, Theorem 1.2] there exists such a bound $n(r, K, g)$, that however also depends on the maximal size g of a subgroup of $\mathcal{S}$. Over a number field, Schur [Sch05] proved that there exists an explicit bound on the size of a finite subgroup of $\mathrm{GL}_d(K)$, so we can bound g independently of $\mathcal{S}$. (See also the, largely expository, article [GL06] for this and later results.) □

Proposition 12.12. *Let $\mathcal{S} \subseteq \overline{\mathbb{Q}}^{d\times d}$ be a finitely generated matrix semigroup, given by a finite set of generators. It is decidable whether or not $\mathcal{S}$ is*

(1) *finite,*

(2) *tame.*

Proof. Let $A_1, \ldots, A_l$ be the given set of generators for $\mathcal{S}$, and let K be the number field generated by all the coefficients of the matrices A_i. Then $\mathcal{S} \subseteq K^{d\times d}$ and it suffices to consider the problem over the field K.

(1) With the bound from Lemma 12.11, one can decide whether or not $\mathcal{S}$ is finite.

(2) This problem can be reduced to the finiteness problem using Lemma 8.5. Indeed, by iterated application of Lemma 12.10, we may decompose $K^{d\times 1} = V_1 \oplus \cdots \oplus V_s$ with each V_i a $\mathcal{S}$-invariant subspace that contains no proper, nonzero $\mathcal{S}$-invariant subspace. Then Lemma 8.5 implies that $\mathcal{S}$ is tame if and only if $\mathcal{S}|_{V_i}$ is finite for each $1 \le i \le s$. This can be decided using (1). □

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