Exercise Sheet 1

Due 8.10.2020

This is a 'warm-up' sheet, exercises are voluntary and count as a bonus.

Problem 1. Let $f, g, h\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$. Which of the following are true?

(a) $2n = O(n)$,

(b) $n = o(5n)$,

(c) for all $\varepsilon > 0$, we have $n = O(\varepsilon n^2)$,

(d) $2n \sim n$,

(e) If $f(n) = O(g(n))$ and $g(n) = o(h(n))$, then $f(n) = o(h(n))$.

(f) We have $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.

Problem 2. Let S be a set of cardinality $n \geq 1$. Show that S has 2^{n-1} subsets of odd cardinality by constructing a bijection between the subsets of odd cardinality and the subsets of even cardinality.

Problem 3. Let $f, g, h\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ be such that

$$f(n) = O(h(n)), \qquad g(n) = O(h(n)), \qquad \text{and } h(n) = o(1).$$

(a) Show that $f(n) + g(n) = O(h(n))$, that $f(n)g(n) = o(h(n))$, and that

$$\frac{1}{1 + f(n)} = 1 + O(h(n)).$$

(b) Use (a) to show

$$\binom{2n}{n} = \frac{4^n}{\sqrt{\pi n}}\Big(1 + O\big(\tfrac{1}{n}\big)\Big).$$

Problem 4. Construct functions $f, g\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $f(n) \neq O(g(n))$ and $g(n) \neq O(f(n))$. Can you make f and g non-decreasing?