Exercise Sheet 4

Due 29.10.2020

Problem 1. A *unary-binary tree* is a plane (unlabeled) tree where each vertex has 0, 1, or 2 descendants. (Recall that in a plane tree the descendants are ordered, e.g., a binary tree is a plane tree where each vertex has 0 or 2 descendants, and if there are descendants, then there is a left and a right descendant.)

(a) Show that the OGF for unary-binary trees is

$$M(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z}.$$

(b) Use Lagrange inversion to show

$$[z^n]M(z) = \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{k-1}.$$

(c) Use singularity analysis to show

$$[z^n]M(z) = 3^n\sqrt{\frac{3}{4\pi n^3}}\left(1 - \frac{15}{16n} + O\Big(\frac{1}{n^2}\Big)\right).$$

Note added 2020/10/26: *Actually the transfer theorems as stated in the lecture notes only allow you to deduce $[z^n]M(z) \sim 3^n\sqrt{\frac{3}{4\pi n^3}}$, so that will be sufficient. Alternatively, use [FS09, Theorem VI.3 or Theorem VI.4].*

Problem 2. Let $\alpha \in \mathbb{C}\setminus\mathbb{Z}_{\le 0}$ and let $f\colon \mathbb{C}\to\mathbb{C}$ be an analytic function. Determine the asymptotic value of $[z^{2n}]f(z^2)(1-z^2)^{-\alpha}$ in two ways:

(a) directly, that is, by applying singularity analysis to $f(z^2)(1-z^2)^{-\alpha}$.

(b) by applying singularity analysis to the function $f(z)(1-z)^{-\alpha}$ and using that $[z^{2n}]f(z^2)(1-z^2)^{-\alpha} = [z^n]f(z)(1-z)^{-\alpha}$.

Problem 3. Denote by $T(z)$ the EGF of rooted labeled trees (we have used this EGF in the proof of Cayley's formula for counting labeled trees) and let $\alpha, \beta \in \mathbb{C}\setminus\{0\}$. Determine $[z^n]\exp(\alpha T(z))$ for $n \ge 0$ and deduce

$$(\alpha+\beta)(n+\alpha+\beta)^{n-1} = \alpha\beta\sum_{k=0}^{n}\binom{n}{k}(k+\alpha)^{k-1}(n-k+\beta)^{n-k-1}$$

for all such α, β, n with $\alpha+\beta \ne 0$.