Exercise Sheet 5
Due 5.11.2020

Problem 1. Show that a graph is bipartite if and only if it does not contain a cycle of odd length.

Problem 2. Use Kőnig's Theorem to deduce Hall's Theorem. That is: Let $G = (A \uplus B, E)$ be a bipartite graph such that $|S| \leq |N(S)|$ for all $S \subseteq A$. Then G contains a matching of A.
Hint: Start by deducing from Kőnig's theorem that G contains a vertex cover $C \subseteq A \cup B$ of cardinality $|C| < |A|$.

Problem 3. Let k, n be positive integers and let X be a set of size kn. Prove that for any two partitions

$$X = \biguplus_{i=1}^{n} U_i \qquad \text{and} \qquad X = \biguplus_{i=1}^{n} V_i \qquad \text{with } |U_i| = |V_i| = k \text{ for all } i \in [n]$$

there exists a common set of representatives $Y \subseteq X$ (that is, $|U_i \cap Y| = |V_i \cap Y| = 1$ for all $i \in [n]$). Show that this is not true if we start with three partitions.

Problem 4. Let G be a bipartite graph. Show that if M is a matching that is not a maximum matching (that is, there exists some matching M' with $|M'| > |M|$), then G contains an augmenting path with respect to M.