Exercise Sheet 7
Due 19.11.2020

Problem 1 (Ramsey's Theorem). Implicit in the definition of the Ramsey numbers $R(k,l)$ in the lecture notes, there is the assumption that for every $k, l \geq 0$ there exists *some* $n \geq 0$ such that every graph on at least n vertices contains a clique of size k or a an independent set of size l. Of course, it is not clear that this is so; indeed this result is the two-colour version of *Ramsey's Theorem*. The goal of this exercise is to prove Ramsey's theorem in an inductive way.

For $k, l \geq 0$ let now $R(k,l)$ be the smallest $n \geq 0$ such that every graph on at least n vertices contains a clique of size k or a an independent set of size l *if there is such an* n. Otherwise we set $R(k,l) = \infty$ and use the convention $n + \infty = \infty + \infty = \infty$ for all $n \geq 0$. Show the following.

(a) $R(k,l) = R(l,k)$ for all $l, k \geq 0$.

(b) $R(0,k) = 0$ and $R(2,k) = k$ for $k \geq 0$; moreover $R(1,k) = 1$ for $k \geq 1$.

(c) $R(k,l) \leq R(k-1,l) + R(k,l-1)$ for all $k, l \geq 2$.

(d) Deduce Ramsey's Theorem: $R(k,l) < \infty$ for all $k, l \geq 0$.

Problem 2. In the lecture we have seen a theorem of Erdős that gives an lower bound for $R(k,k)$ that is exponential in k. The proof used the probabilistic method and did not actually construct a family of graphs attaining that lower bound (indeed, it is not known how do to so!). Construct a family of graphs that shows

$$R(k,k) \geq (k-1)^2.$$

(This is, of course, a much weaker lower bound!)

Problem 3. Let $(G,+)$ be an additive group. A subset $S \subseteq G$ is called *sum-free* if there exist no $a, b, c \in S$ with $a + b = c$. Let $A \subseteq \mathbb{Z} \setminus \{0\}$ be a finite, non-empty subset. Show that there exists a sum-free subset $B \subseteq A$ with $|B| > |A|/3$.

To do so, make use of the probabilistic method. The following are the main steps.

(a) Choose a prime p of the form $p = 3k + 2$ such that A is sum-free if and only if the image of $\overline{A}$ of A in $\mathbb{Z}/p\mathbb{Z}$ is sum-free.

(b) Note that $Y := \{\overline{y} \in \mathbb{Z}/p\mathbb{Z} : y \in \mathbb{Z},\ k+1 \leq y \leq 2k+1\}$ is sum-free in $\mathbb{Z}/p\mathbb{Z}$.

(c) Consider $B = \overline{A} \cap xY$ with $x \in (\mathbb{Z}/p\mathbb{Z})^\times$ chosen uniformely at random, and show that there is such a set with size $|B| > |A|/3$ with positive probability.