Exercise Sheet 8

Due 26.11.2020

Problem 1. Let $G = (V, E)$ be a graph.

(a) Suppose that G contains a matching M consisting of m edges. Prove that there exists a set $S \subseteq V$ such that there are at least

$$\frac{|E| + m}{2}$$

edges between S and $V \setminus S$.

(b) Suppose that $|E| \leq c^2$ for some fixed $c \in \mathbb{N}$. Prove that it is possible to assign a colour to each vertex of G so that at most $2c$ different colours are used in total and for each edge, its end vertices have different colours.

Note/Hint. Both parts can be proven by determining the expectation of a suitable random variable. For (a), choosing S randomly among all subsets of V would only suffice to obtain $|E|/2$ edges. For (b), first consider a random colouring using only c colours.

Problem 2. Let X be a random variable and $t > 0$. Prove Chebychev's inequality,

$$\mathbb{P}\big[|X - \mathbb{E}[X]| \geq t\big] \leq \frac{\mathrm{Var}[X]}{t^2}.$$

Problem 3. Prove that the property $\omega(G) \geq 4$ has threshold function $n^{-2/3}$ (in the binomial random graph model) in two ways:

(a) directly, along the lines of the proof of Theorem 3.3.10 (Triangle threshold);

(b) by applying the more general Theorem 3.3.12 (Subgraph threshold).