Exercise Sheet 9

Due 3.12.2020

Throughout, vector spaces are considered over an arbitrary but fixed field K.

Problem 1. Let U, V be vector spaces.

(a) Let $u, u' \in U$, $v, v' \in V$ and $\lambda \in K$. Show that $\lambda(u \otimes v) = (\lambda u) \otimes v = u \otimes (\lambda v)$, that $(u + u') \otimes v = u \otimes v + u' \otimes v$, and that $u \otimes (v + v') = u \otimes v + u \otimes v'$. (*Note*: this is trivial if you understand the definitions.)

(b) Let $x \in U \otimes V$. Suppose that $k \geq 0$ is minimal such that there exist $u_1, \dots, u_k \in U$ and $v_1, \dots, v_k \in V$ with $x = \sum_{i=1}^{k} u_i \otimes v_i$. Show that $(u_1, \dots, u_k)$, respectively $(v_1, \dots, v_k)$, are linearly independent.

Problem 2. Let $V_1, \dots, V_m, W, P$ be vector spaces, and let $\varphi\colon V_1 \times \cdots \times V_m \to P$ be a tensor map. If $\psi\colon V_1 \times \cdots V_m \to W$ is a multilinear map, then there exists a linear map $T\colon P \to W$ such that $T \circ \varphi = \psi$. Show that T is unique if and only if $\langle \operatorname{im} \varphi \rangle = P$.

Problem 3. Let U, V, W be vector spaces. Show that there are natural[1] isomorphisms $U \otimes V \cong V \otimes U$ satisfying $u \otimes v \mapsto v \otimes u$ and $(U \otimes V) \otimes W \cong U \otimes V \otimes W$ satisfying $(u \otimes v) \otimes w \mapsto u \otimes v \otimes w$.

Problem 4. Let U, V, W be vector spaces. Show that there are natural isomorphisms

$$\operatorname{Hom}(U, \operatorname{Hom}(V, W)) \cong M(U, V; W) \quad \text{and} \quad M(U, V; W) \cong \operatorname{Hom}(U \otimes V, W).$$

(Here $M(U, V; W)$ denotes the vector space of bilinear maps $U \times V \to W$.)

[1] We use the word *natural* in an informal way here. It is easy to see that these spaces are isomorphic by comparing their dimension, but the goal is to construct explicit isomorphisms. The word *natural* in this context can be given a precise meaning in the context of category theory: the isomorphisms must be functorial in U, V, and W.