Exercise Sheet 4
Due 29.10.2020

Problem 1. A *unary-binary tree* is a plane (unlabeled) tree where each vertex has 0, 1, or 2 descendants. (Recall that in a plane tree the descendants are ordered, e.g., a binary tree is a plane tree where each vertex has 0 or 2 descendants, and if there are descendants, then there is a left and a right descendant.)

(a) Show that the OGF for unary-binary trees is

$$M(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z}.$$

(b) ...

(c) Use singularity analysis to show

$$[z^n]M(z) = 3^n \sqrt{\frac{3}{4\pi n^3}} \left(1 - \frac{15}{16n} + O\Big(\frac{1}{n^2}\Big)\right).$$

Solution to (c) First note that $M(z)$, as stated, has a singularity at 0. However, since $\lim_{z\to 0} M(z)$ exists, the singularity at 0 is removable (e.g., use Riemann's theorem on removable singularities [1]). After continuing $M(z)$ analytically to 0, the dominant singularity is $\frac{1}{3}$. Thus $M(z)$ may be extended analytically to the open disc $B(0, \frac{1}{3})$ centered at 0 with radius $\frac{1}{3}$. Since the term $\sqrt{(1+z)(1-3z)}$ is analytic in $\mathbb{C}\setminus\big((-\infty, -1] \cup [\frac{1}{3}, \infty)\big)$, the same is true for $M(z)$. Thus there exists a Δ-domain $\Delta_0 = \Delta_0(r, \varphi)$ such that $M(z)$ is analytic in $\frac{1}{3}\Delta_0$ (here $r > 0$, $\varphi \in (0, \pi/2)$ can be chosen arbitrarily).

At this point we write $M(z) = f(z) - g(z)\sqrt{1-3z}$ with $f(z) = \frac{1-z}{2z}$ and $g(z) = \frac{\sqrt{1+z}}{2z}$. Since both $f(z)$ and $g(z)$ are analytic at $1/3$, we obtain Taylor series expansions

$$\begin{aligned} f(z) &= 1 + \tfrac{3}{2}(1-3z) + O((1-3z)^2) \\ g(z) &= \sqrt{3} + \tfrac{7}{8}\sqrt{3}(1-3z) + O((1-3z)^2). \end{aligned}$$

(If we just want $\sim$-growth, of course fewer terms are needed.) This gives a singular expansion $M(z) = S(z) + O((1-3z)^2)$ with

$$S(z) = 1 - \sqrt{3}(1-3z)^{1/2} + \tfrac{3}{2}(1-3z) - \tfrac{7}{8}\sqrt{3}(1-3z)^{3/2}.$$

Now we want to apply Theorem 1.4.20 (“Big-Oh, Little-Oh Transfer”). For purposes of illustration we do it very explicitly once. Note that $M(z) - S(z)$ is analytic in $\frac{1}{3}\Delta_0$.

[1] Another way: Expand the numerator as power series at 0 and note that the constant term vanishes. Therefore the quotient is still analytic.

Therefore $M(\frac{z}{3}) - S(\frac{z}{3})$ is Δ-analytic and therefore satisfies the assumptions of the theorem. We find[2]

$$[z^n]\big(M(\tfrac{z}{3}) - S(\tfrac{z}{3})\big) = O(n^{-3}).$$

Therefore

$$[z^n]M(z) = 3^n M(\tfrac{z}{3}) = 3^n S(\tfrac{z}{3}) + O(n^{-3}).$$

The problem is therefore reduced to determining the asymptotic growth of $[z^n]S(z/3)$ to sufficient order. For $\sim$-growth, it suffices to note $[z^n]S(z/3) \sim -\sqrt{3}n^{-3/2}\Gamma(-1/2)^{-1}$ (coming from the $-\sqrt{3}(1-3z)^{1/2}$-term). Since $z\Gamma(z) = \Gamma(z+1)$, we have $\Gamma(-1/2) = -2\Gamma(1/2) = -2\sqrt{\pi}$. Thus

$$[z^n]M(z) \sim 3^n\sqrt{3}\frac{1}{\sqrt{4\pi n^3}}.$$

(Note that the polynomial summands don't matter, because they only have finally many nonzero coefficients. Explicitly, for $p(z)$ a polynomial we have $[z^n]p(z) = O(n^{-k})$ for all $k \geq 1$.)

To obtain additional terms of the asymptotic expansion, we need better asymptotics on $[z^n](1-z)^{-\alpha}$ in the case $\alpha = -1/2$. We refer to [FS09, Theorem VI.1] or [FS09, Figure VI.5 on p.372] to find, e.g.,

$$\begin{aligned}
(1-z)^{1/2} &= -\frac{1}{\sqrt{\pi n^3}}\left(\frac{1}{2} + \frac{3}{16n} + O(n^{-2})\right),\\
(1-z)^{3/2} &= \frac{1}{\sqrt{\pi n^5}}\left(\frac{3}{4} + O(n^{-1})\right).
\end{aligned}$$

Thus, substituting into the formula for $S(z)$,

$$[z^n]M(z) = 3^n\sqrt{3}\left[\frac{1}{\sqrt{\pi n^3}}\left(\frac{1}{2} + \frac{3}{16n} + O(n^{-2})\right) - \frac{7}{8}\frac{1}{\sqrt{\pi n^5}}\left(\frac{3}{4} + O(n^{-1})\right)\right]$$

And, gathering terms,

$$M(z) = \frac{3^n\sqrt{3}}{\sqrt{4\pi n^3}}\left[1 - \frac{15}{16n} + O(n^{-2})\right].$$

Clearly, additional terms can be computed by (i) computing the Taylor series of $f(z)$, $g(z)$ to higher order; and (ii) computing coefficient-asymptotics of $(1-z)^{-\alpha}$ for $\alpha = -1/2$, $-3/2$, $-5/2, \ldots$, to sufficient order.

[2] Technically, the theorem as stated in the notes is not applicable for $\alpha = -2$. But the first part still holds; alternatively, replace $O((1-3z)^2)$ by $O((1-3z)^{2-\varepsilon})$ for arbitrarily small $\varepsilon > 0$...