Problem 32. The OGF $M(z)$ of unary-binary trees satisfies

$$M(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z} \qquad \text{(cf. Problem 31).}$$

Use singularity analysis to show that

$$[z^n]M(z) = 3^n\sqrt{\frac{3}{4\pi n^3}}\left(1 + O\left(\frac{1}{n}\right)\right).$$

Voluntarily: You can also try to show that, more precisely,

$$[z^n]M(z) = 3^n\sqrt{\frac{3}{4\pi n^3}}\left(1 + \frac{15}{16n} + O\left(\frac{1}{n^2}\right)\right).$$

For this you will need more precise asymptotics for the coefficients of $(1-z)^{-\alpha}$ with $\alpha = -1/2$, than were discussed in the lecture. One can derive such asymptotics to any desired accuracy (see [FS09, Theorem VI.1] and [FS09, Figure VI.5 on p.372]). In particular, for the refined asymptotics of $M(z)$ one needs

$$(1-z)^{1/2} = -\frac{1}{\sqrt{\pi n^3}}\left(\frac{1}{2} + \frac{3}{16n} + O(n^{-2})\right).$$

Problem 33. Use singularity analysis to determine the asymptotics of

$$\frac{e^{-z}(1+2z)^{3/2}}{\sqrt{1-2z}}.$$